A Reduction-based Cut-free Gentzen Calculus for Dynamic Epistemic Logic

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Dedicated to John N. Crossley

Abstract

Dynamic Epistemic Logic is a multi-modal logic for reasoning about the change of knowledge in multi-agent systems. It extends epistemic logic by a modal operator for actions which announce logical formulas to other agents. In Hilbert-style proof calculi for Dynamic Epistemic Logic modal action formulas are reduced to epistemic logic, whereas current sequent calculi for Dynamic Epistemic Logic are labelled systems which internalise the semantic accessibility relation of the modal operators as well as the accessibility relation underlying the semantics of the actions. We present a novel cut-free ordinary sequent calculus, called $\mathbf{G4}_{P,A}[]$, for propositional Dynamic Epistemic Logic. In contrast to the known sequent calculi, our calculus does not internalise the accessibility relations, but — similar to Hilbert style proof calculi — action formulas are reduced to epistemic formulas. Since no ordinary sequent calculus for full **S5** modal logic is known, the proof rules for the knowledge operator and the Boolean operators are those of an underlying **S4** modal calculus. We show the soundness and completeness of $\mathbf{G4}_{P,A}[]$ and prove also the admissibility of the cut-rule and of several other rules for introducing the action modality.

Keywords: Dynamic Epistemic Logic; Sequent Calculus; Cut Elimination

1 Introduction

Dynamic Epistemic Logic (DEL) is a framework for reasoning about the change of knowledge in multi-agent systems. It is based on epistemic logic, a multi-modal logic in which the modal operators express the knowledge and the belief of the agents. The main additional feature of DEL is the communication of epistemic information. Using so-called (epistemic) actions, agents can send public, private, and semi-private announcements to one or more agents. In the logic this is expressed by a modal operator $[\mathfrak{u}]$ for epistemic actions $\mathfrak{u}$ and formulæ of the form $[\mathfrak{u}]\psi$ with the meaning that always after executing the action $\mathfrak{u}$, the formula ψ holds. Public Announcement Logic (PAL), a simplified variant of DEL, restricts actions to public announcements.

There exist several proof calculi for DEL and PAL. Sound and complete Hilbert-style axiomatisations are given for PAL by Plaza [1, 2] and for DEL by Baltag, Moss, and Solecki [3] and Gerbrandy [4] (see [5] for an overview). These proof systems are based on Hilbert calculi for epistemic logic and translate modal formulæ of the form $[\mathfrak{u}]\psi$ into pure epistemic logic formulæ without announcement actions. For PAL and DEL also tableaux and display calculi have been developed (Balbiani et al. [6], Hansen [7], Aucher et al. [8], Frittella et al. [9], for a comparison with other proof systems see Frittella et al. [10]). A first sequent calculus for DEL has been presented by Baltag et al. [11]. Actions enjoy a quantal structure; propositions, actions, and agents are resource-sensitive. The calculus is sound and complete but does not admit the elimination of cuts. Dyckhoff and Sadrzadeh [12] refine this proof calculus to a cut-free calculus. However, this calculus does not use ordinary sequents but more complex nested sequents.

This is similar to the situation in modal logic. Only for modal logic systems **S4** or smaller, ordinary sequent calculi are known which are sound, complete, and cut-free. For modal logic **S5**, such proof systems need global side conditions (as in Braüner [13] or extend the sequence format by additional structure such as hypersequents (see, e. g., Poggiolesi [14]) and display systems (see, e. g., Dosen [15]); for an overview cf. Wansing [16] and Negri [17]. Labelled sequent systems internalise the Kripke semantics of modal logic into the syntax of the proof system. Such calculi are ordinary sequent systems which not only contain modalities but also variables and the semantic accessibility relation (see, e. g., Brünnler [18]). Negri [19] presents a general method for generating contraction- and cut-free ordinary sequent calculi for a large family of normal modal logics. Her method has been applied by several authors for constructing labelled cut-free sequent calculi for public announcement logic (see Maffezioli and Negri [20], Negri [17], Balbiani et al. [21], Nomura et al. [22], and Balbiani and Galmiche [23]). The cut-free sequent calculus of Nomura et al. [24, 25] for full DEL internalises the semantic accessibility relation of the modal operators as well as the accessibility relation underlying the semantics of the actions.

We present a novel cut-free sequent calculus, called $\mathbf{G4}_{P,A}[]$, for propositional Dynamic Epistemic Logic. In contrast to the labelled sequent calculi, our calculus does not internalise the accessibility relations nor does it contain labels, instead the rules for epistemic actions mirror the reduction rules of [5, 26]; these rules are invertible, but do not enjoy the subterm property. As underlying modal system we choose an **S4** calculus, since no ordinary sequent calculus for full **S5** modal logic is not known. We show the soundness and completeness of $\mathbf{G4}_{P,A}[]$ and prove also the admissibility of the cut-rule and of several rules for introducing the action modality. Neither for completeness nor for the cut we apply the well-known translation of [5], instead we give direct proofs of the admissibility of cut and of all axioms and rules of the Hilbert calculus for $\mathbf{DS4}_{P,A}$. Closely related to our work is the independently developed labelled sequent calculus of Wu et al. [27] for PAL. Similar to our approach, the proof rules of [27] follow the structure of the goal and reduce (PAL) formulas to basic epistemic logic formulas. But in contrast to us, the semantic accessibility relation is internalised and the proofs of completeness and admissibility of cut use the translation to epistemic logic.

The paper is organised as follows: In Sect. 2 we recap the basics of epistemic logic and present the sequent calculus $\mathbf{G4}_{P,A}$ together with some derived rules and the main theorems for soundness, completeness, and admissibility of cut. Section 3 contains the main results: We present the ordinary sequent calculus $\mathbf{G4}_{P,A}[]$ for DEL, show some derived rules including a particular kind of necessitation for dynamic modalities, and prove soundness and completeness of $\mathbf{G4}_{P,A}[]$ and the admissibility of cuts. Section 4 concludes with an outlook to future work.

Personal note. John, Martin, and Alexander have known each other for many years. Alexander met John for the first time at the end of the 1990s in the Research Training Group "Logic in Computer Science"[1] when he was a PhD student in the group and John a guest researcher. Martin and John had met much earlier, in 1975 at a garden party of Martin's doctoral supervisor Kurt Schütte. The day after the party, John, Martin and the logician Peter Päppinghaus drove together in Martin's car to the "Colloque International de Logique" in Clermont-Ferrand. They became good friends, though communication was difficult. Although each of the three spoke two languages, there was no common language: John spoke English and French, Martin German and French, and Peter German and English.

About 10 years later, a close collaboration developed between John and Martin. John visited Martin regularly in Passau and later in Munich, Martin was twice in Melbourne with John in the late 1990s. Together with their students, they worked on two research topics, the development of constrained λ-calculi and program extraction from structured specifications. Four papers [28, 29, 30, 31] were written on the first topic, as well as the dissertations of Luis Mandel [32] and Matthias Hölzl [33]. Luis and Matthias were also jointly supervised by Martin and John. On the second topic, John and Martin wrote three papers [34, 35, 36] together with Hannes Peterreins and Iman Poernomo, a doctoral student of John. An important part of the joint monograph [37] also deals with this topic. At that time, Alexander worked on other topics including the semantics of Java [38] and formal approaches to mobile systems [39] and object-oriented software development [40].

Working and discussing with John is a very pleasant experience. He is not only an outstanding scientist; he is also a warm-hearted and kind friend and colleague. We are looking forward to many further inspiring exchanges with him.

2 Epistemic Logic

Propositional epistemic logic is a multi-modal logic. We briefly recall some basic definitions and results about Gentzen type proof systems.

An *epistemic signature* (P, A) consists of a set P of *propositions* and a set A of *agents*. The set $\Phi_{P,A}$ of *epistemic formulæ* φ over (P, A) is defined by the following grammar:

$$\varphi ::= p \mid \text{false} \mid \varphi_1 \supset \varphi_2 \mid \mathsf{K}_a\, \varphi$$

where $p \in P$ and $a \in A$. The epistemic formula $\mathsf{K}_a\, \varphi$ is to be read as "agent a *knows* φ". The usual propositional connectives can be added by defining $\neg\varphi \equiv \varphi \supset \text{false}$, $\varphi_1 \vee \varphi_2 \equiv (\neg\varphi_1) \supset \varphi_2$, $\varphi_1 \wedge \varphi_2 \equiv \neg(\varphi \supset \neg\varphi_2)$, and $\varphi_1 \leftrightarrow \varphi_2 \equiv (\varphi_1 \supset \varphi_2) \wedge (\varphi_2 \supset \varphi_1)$.

An *epistemic* (**S4**) *structure* $K = (W, E, L)$ over (P, A) consists of a set W of *worlds*, an A-indexed family $E = (E_a \subseteq W \times W)_{a \in A}$ of epistemic *accessibility relations*, and a *labelling* $L \colon W \to \mathcal{P}(P)$ which determines

[1] https://gepris.dfg.de/gepris/projekt/271709?language=en

$$\text{(taut) propositional tautologies} \qquad \text{(K) } \mathsf{K}_a(\varphi_1 \supset \varphi_2) \supset (\mathsf{K}_a \varphi_1 \supset \mathsf{K}_a \varphi_2)$$

$$\text{(T) } \mathsf{K}_a \varphi \supset \varphi \qquad \text{(4) } \mathsf{K}_a \varphi \supset \mathsf{K}_a \mathsf{K}_a \varphi$$

$$\text{(MP) } \frac{\varphi_1 \quad \varphi_1 \supset \varphi_2}{\varphi_2} \qquad \text{(GK) } \frac{\varphi}{\mathsf{K}_a \varphi}$$

Table 1: Hilbert-style axiomatisation of $\mathbf{S4}_{P,A}$

$$(p\mathrm{A})\ \frac{}{p, \Gamma \Rightarrow p, \Delta}$$

$$(\mathrm{Lfalse})\ \frac{}{\mathrm{false}, \Gamma \Rightarrow \Delta}$$

$$(\mathrm{L}{\supset})\ \frac{\Gamma \Rightarrow \varphi_1, \Delta \quad \varphi_2, \Gamma \Rightarrow \Delta}{\varphi_1 \supset \varphi_2, \Gamma \Rightarrow \Delta} \qquad (\mathrm{R}{\supset})\ \frac{\varphi_1, \Gamma \Rightarrow \varphi_2, \Delta}{\Gamma \Rightarrow \varphi_1 \supset \varphi_2, \Delta}$$

$$(\mathrm{LT})\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\mathsf{K}_a \varphi, \Gamma \Rightarrow \Delta} \qquad (\mathrm{R}\mathsf{K})\ \frac{\mathsf{K}_a \Gamma \Rightarrow \varphi}{\mathsf{K}_a \Gamma, \Gamma' \Rightarrow \mathsf{K}_a \varphi, \Delta'}$$

Table 2: Modal Gentzen system $\mathbf{G4}_{P,A}$ for epistemic logic $\mathbf{S4}_{P,A}$

$$(\mathrm{Weak})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \qquad (\mathrm{Contr})\ \frac{\Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

$$(\mathrm{Cut})\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Table 3: Structural rules and cut

for each world $w \in W$ the set of propositions valid in w. The accessibility relations of epistemic structures are assumed to be reflexive and transitive (but not necessarily symmetric as in **S5**). For any $a \in A$, $(w, w') \in E_a$ models that agent a cannot distinguish the two worlds w and w'. An *epistemic* (**S4**) *state* over (P, A) is a pointed epistemic structure $\mathfrak{K} = (K, w)$ where $w \in W$ determines an actual world.

For any epistemic signature (P, A) and epistemic structure $K = (W, E, L)$ over (P, A) the *satisfaction* of an epistemic formula $\varphi \in \Phi_{P,A}$ by K at a world $w \in W$, written $K, w \models \varphi$, is inductively defined as follows for any $a \in A$, $p \in P$, and $\varphi, \varphi_1, \varphi_2 \in \Phi_{P,A}$:

$$K, w \models p \iff p \in L(w)$$
$$K, w \not\models \mathrm{false}$$
$$K, w \models \varphi_1 \supset \varphi_2 \iff K, w \not\models \varphi_1 \text{ or } K, w \models \varphi_2$$
$$K, w \models \mathsf{K}_a \varphi \iff K, w' \models \varphi \text{ for all } w' \in W \text{ with } (w, w') \in E_a$$

Hence, an agent a knows φ at world w if φ holds in all worlds w' which a cannot distinguish from w. For an epistemic state $\mathfrak{K} = (K, w)$ and for $\varphi \in \Phi_{P,A}$, $\mathfrak{K} \models \varphi$ means $K, w \models \varphi$.

The epistemic logic $\mathbf{S4}_{P,A}$ consists of all epistemic formulæ $\varphi \in \Phi_{P,A}$ such that $K, w \models \varphi$ for all epistemic structures $K = (W, E, L)$ and all their states $w \in W$. This logic can be axiomatised in a Hilbert-calculus by the axioms and derivation rules of Tab. 1 (see, e. g., [5]) where axiom (T), called *truth*, reflects the reflexivity of the accessibility relations and axiom (4), called *positive introspection*, their transitivity.

We use the modal Gentzen system $\mathbf{G4}_{P,A}$ in Tab. 2 for the epistemic logic $\mathbf{S4}_{P,A}$. Our system builds on $\mathbf{G3}_n\mathbf{K}$ for basic modal logic (Hakli and Negri [41]), and for the extension to **S4** on the system $\mathbf{S4}^*$ (Ohnishi and Matsumoto [42]) and the system **GS4** (Ono [43]). In our rules, $\varphi, \varphi_1, \varphi_2$ range over the formulæ in $\Phi_{P,A}$, p over the propositions in P, a over the agents in A, and $\Gamma, \Gamma', \Delta, \Delta'$ over the multisets of formulæ in $\Phi_{P,A}$. In particular, Γ can be empty in (RK), i. e., this multiset can be dropped; then (RK) is a direct generalisation of (GK).

Lemma 1. *All sequents of the form $\varphi, \Gamma \Rightarrow \Delta, \varphi$ are derivable in $\mathbf{G4}_{P,A}$.*

Proof. By structural induction over φ, see, e. g., [41]. $\square$

The structural rules, see Tab. 3, of *weakening* and *contraction* are admissible, and so is *cut*.

$$(\mathrm{L}\neg)\ \frac{\Gamma \Rightarrow \varphi, \Delta}{\neg\varphi, \Gamma \Rightarrow \Delta} \qquad (\mathrm{R}\neg)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta}$$

$$(\mathrm{L}\vee)\ \frac{\varphi_1, \Gamma \Rightarrow \Delta \quad \varphi_2, \Gamma \Rightarrow \Delta}{\varphi_1 \vee \varphi_2, \Gamma \Rightarrow \Delta} \qquad (\mathrm{R}\vee)\ \frac{\Gamma \Rightarrow \varphi_1, \varphi_2, \Delta}{\Gamma \Rightarrow \varphi_1 \vee \varphi_2, \Delta}$$

$$(\mathrm{L}\wedge)\ \frac{\varphi_1, \varphi_2, \Gamma \Rightarrow \Delta}{\varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Delta} \qquad (\mathrm{R}\wedge)\ \frac{\Gamma \Rightarrow \varphi_1, \Delta \quad \Gamma \Rightarrow \varphi_2, \Delta}{\Gamma \Rightarrow \varphi_1 \wedge \varphi_2, \Delta}$$

$$(\mathrm{L}\leftrightarrow)\ \frac{\varphi_1, \varphi_2, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \varphi_1, \varphi_2, \Delta}{\varphi_1 \leftrightarrow \varphi_2, \Gamma \Rightarrow \Delta} \qquad (\mathrm{R}\leftrightarrow)\ \frac{\varphi_1, \Gamma \Rightarrow \varphi_2, \Delta \quad \varphi_2, \Gamma \Rightarrow \varphi_1, \Delta}{\Gamma \Rightarrow \varphi_1 \leftrightarrow \varphi_2, \Delta}$$

$$(\mathrm{RT})\ \frac{\Gamma \Rightarrow \mathsf{K}_a \varphi, \Delta}{\Gamma \Rightarrow \varphi, \Delta}$$

$$(\mathsf{LK}^2)\ \frac{\mathsf{K}_a \mathsf{K}_a \varphi, \Gamma \Rightarrow \Delta}{\mathsf{K}_a \varphi, \Gamma \Rightarrow \Delta} \qquad (\mathsf{RK}^2)\ \frac{\Gamma \Rightarrow \mathsf{K}_a \varphi, \Delta}{\Gamma \Rightarrow \mathsf{K}_a \mathsf{K}_a \varphi, \Delta}$$

$$(\mathsf{LRK})\ \frac{\Gamma \Rightarrow \varphi}{\mathsf{K}_a \Gamma, \Gamma' \Rightarrow \mathsf{K}_a \varphi, \Delta'}$$

Table 4: Additional rules for $\mathbf{G4}_{P,A}$

Lemma 2. (Weak) *and* (Contr) *are height-preservingly admissible for* $\mathbf{G4}_{P,A}$.

Proof. By induction on the height of the derivation (as in, e. g., [44]). ☐

Theorem 1. (Cut) *is admissible for* $\mathbf{G4}_{P,A}$.

Proof. As in, e. g., [43]. ☐

Theorem 2. $\mathbf{G4}_{P,A}$ *is sound and complete for* **S4**, *i. e., for any* $\varphi \in \Phi_{P,A}$, $\vdash_{\mathbf{G4}_{P,A}} \Rightarrow \varphi$ *if, and only if,* $\mathfrak{K} \models \varphi$ *in all epistemic* **S4** *states* $\mathfrak{K}$ *over* (P, A).

Proof. For soundness, it suffices to check that each rule of $\mathbf{G4}_{P,A}$ is valid in $\mathbf{S4}_{P,A}$; for completeness, that each axiom of the Hilbert-style axiomatisation in Tab. 1 is derivable in $\mathbf{G4}_{P,A}$ and that each rule is admissible. ☐

Table 4 contains derived rules for the other propositional connectives. Additionally, it shows admissible rules for truth, (RT), and positive introspection, (LK^2), and (RK^2):

Lemma 3. *For all* $a \in A$, *all* $\varphi \in \Phi_{P,A}$, *and all multisets* Γ, Δ *of formulæ the following statements hold:*

(a) If $\vdash_{\mathbf{G4}_{P,A}} \Gamma \Rightarrow \mathsf{K}_a \varphi, \Delta$, *then* $\vdash_{\mathbf{G4}_{P,A}} \Gamma \Rightarrow \varphi, \Delta$.
(b) If $\vdash_{\mathbf{G4}_{P,A}} \mathsf{K}_a \mathsf{K}_a \varphi, \Gamma \Rightarrow \Delta$, *then* $\vdash_{\mathbf{G4}_{P,A}} \mathsf{K}_a \varphi, \Gamma \Rightarrow \Delta$.
(c) If $\vdash_{\mathbf{G4}_{P,A}} \Gamma \Rightarrow \mathsf{K}_a \varphi, \Delta$, *then* $\vdash_{\mathbf{G4}_{P,A}} \Gamma \Rightarrow \mathsf{K}_a \mathsf{K}_a \varphi, \Delta$.

Proof. For all claims we proceed by induction over the derivation of the premiss and consider the last rule applied. The cases (pA), (Lfalse), (L$\supset$), (R$\supset$) are immediate since neither $\mathsf{K}_a \varphi$ in the succedent nor $\mathsf{K}_a \mathsf{K}_a \varphi$ in the antecedent is principal in these rules; the same holds for (a) and (c) with (LT), where $\mathsf{K}_a \varphi$ is in the succedent.

(a) We only consider (RK). Then $\Gamma = \mathsf{K}_{a'} \Gamma', \Gamma''$ and $\mathsf{K}_a \varphi, \Delta = \mathsf{K}_{a'} \varphi', \Delta'$ for some $a', \Gamma', \Gamma'', \varphi', \Delta'$, and $\vdash_{\mathbf{G4}_{P,A}} \mathsf{K}_{a'} \Gamma' \Rightarrow \varphi'$. If $\mathsf{K}_a \varphi$ is principal, i. e., $\mathsf{K}_a \varphi = \mathsf{K}_{a'} \varphi'$ and $\Delta = \Delta'$, then $\vdash_{\mathbf{G4}_{P,A}} \Gamma \Rightarrow \varphi, \Delta$ follows from weakening $\vdash_{\mathbf{G4}_{P,A}} \mathsf{K}_a \Gamma' \Rightarrow \varphi$. If $\mathsf{K}_a \varphi$ is not principal, i. e., $\Delta = \mathsf{K}_{a'} \varphi', \Delta''$, then $\vdash_{\mathbf{G4}_{P,A}} \mathsf{K}_{a'} \Gamma', \Gamma'' \Rightarrow \mathsf{K}_{a'} \varphi', \varphi, \Delta''$ by applying (RK) with premiss $\mathsf{K}_{a'} \Gamma' \Rightarrow \varphi'$, that is, $\vdash_{\mathbf{G4}_{P,A}} \Gamma \Rightarrow \varphi, \Delta$.

(b) We only consider (LT) and (RK).

Case (LT): Then immediately $\vdash_{\mathbf{G4}_{P,A}} \mathsf{K}_a \varphi, \Gamma \Rightarrow \Delta$.

Case (RK): Then $\mathsf{K}_a \mathsf{K}_a \varphi, \Gamma = \mathsf{K}_{a'} \Gamma', \Gamma''$ and $\Delta = \mathsf{K}_{a'} \varphi', \Delta'$ for some $a', \Gamma', \Gamma'', \varphi', \Delta'$. If $a = a'$, then $\vdash_{\mathbf{G4}_{P,A}} \mathsf{K}_a \mathsf{K}_a \varphi, \mathsf{K}_a \Gamma' \Rightarrow \varphi'$, thus $\vdash_{\mathbf{G4}_{P,A}} \mathsf{K}_a \varphi, \mathsf{K}_a \Gamma' \Rightarrow \varphi'$ by the induction hypothesis, and hence $\vdash_{\mathbf{G4}_{P,A}} \mathsf{K}_a \varphi, \mathsf{K}_a \Gamma', \Gamma'' \Rightarrow \mathsf{K}_a \varphi', \Delta'$ using (RK). If $a \neq a'$, then $\vdash_{\mathbf{G4}_{P,A}} \mathsf{K}_{a'} \Gamma' \Rightarrow \varphi'$ and thus $\vdash_{\mathbf{G4}_{P,A}} \mathsf{K}_a \varphi, \mathsf{K}_a \Gamma', \Gamma'' \Rightarrow \mathsf{K}_a \varphi', \Delta'$ again by (RK).

(c) We only consider (RK). Then $\Gamma = \mathsf{K}_{a'}\,\Gamma', \Gamma''$ and $\mathsf{K}_a\,\varphi, \Delta = \mathsf{K}_{a'}\,\varphi', \Delta'$ for some $a', \Gamma', \Gamma'', \varphi', \Delta'$, and $\vdash_{\mathbf{G4}_{P,A}} \mathsf{K}_{a'}\,\Gamma' \Rightarrow \varphi'$. If $\mathsf{K}_a\,\varphi$ is principal, i. e., $\mathsf{K}_a\,\varphi = \mathsf{K}_{a'}\,\varphi'$ and $\Delta = \Delta'$, then $\vdash_{\mathbf{G4}_{P,A}} \mathsf{K}_a\,\Gamma' \Rightarrow \varphi$, such that $\vdash_{\mathbf{G4}_{P,A}} \mathsf{K}_a\,\Gamma' \Rightarrow \mathsf{K}_a\,\mathsf{K}_a\,\varphi$ by applying (RK) twice, which yields $\vdash_{\mathbf{G4}_{P,A}} \Gamma \Rightarrow \mathsf{K}_a\,\mathsf{K}_a\,\varphi, \Delta$ by weakening. If $\mathsf{K}_a\,\varphi$ is not principal, i. e., $\Delta = \mathsf{K}_{a'}\,\varphi', \Delta''$ for some Δ'', then $\vdash_{\mathbf{G4}_{P,A}} \mathsf{K}_{a'}\,\Gamma', \Gamma'' \Rightarrow \mathsf{K}_{a'}\,\varphi', \mathsf{K}_a\,\mathsf{K}_a\,\varphi, \Delta'$ by applying (RK) with premiss $\mathsf{K}_{a'}\,\Gamma' \Rightarrow \varphi'$, that is, $\vdash_{\mathbf{G4}_{P,A}} \Gamma \Rightarrow \mathsf{K}_a\,\mathsf{K}_a\,\varphi, \Delta$. □

The asymmetric rule (RK) may be replaced by a more symmetric variant (LRK) if not only the truth rule (LT) but also the positive introspection rule (LK^2) is present:

Lemma 4. *If $\vdash_{\mathbf{G4}_{P,A}} \Gamma \Rightarrow \varphi$, then $\vdash_{\mathbf{G4}_{P,A}} \mathsf{K}_a\,\Gamma, \Gamma' \Rightarrow \mathsf{K}_a\,\varphi, \Delta'$. Conversely, replace* (RK) *by* (LRK) *and call the resulting system $\mathbf{G4}'_{P,A}$: If $\vdash_{\mathbf{G4}'_{P,A}} \mathsf{K}_a\,\Gamma \Rightarrow \varphi$, then $\vdash_{\mathbf{G4}'_{P,A}} \mathsf{K}_a\,\Gamma, \Gamma' \Rightarrow \mathsf{K}_a\,\varphi, \Delta'$.*

Proof. In $\mathbf{G4}_{P,A}$ we have the following derivation to the left, for the converse direction using $\mathbf{G4}'_{P,A}$ the derivation to the right, where $(\mathsf{LT})^*$ and $(\mathsf{LK}^2)^*$ mean an iterated rule application (including zero iterations):

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow \varphi\end{matrix}}{\mathsf{K}_a\,\Gamma \Rightarrow \varphi}\,(\mathsf{LT})^*}{\mathsf{K}_a\,\Gamma, \Gamma' \Rightarrow \mathsf{K}_a\,\varphi, \Delta'}\,(\mathsf{RK}) \qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ \mathsf{K}_a\,\Gamma \Rightarrow \varphi\end{matrix}}{\mathsf{K}_a\,\mathsf{K}_a\,\Gamma, \Gamma' \Rightarrow \mathsf{K}_a\,\varphi, \Delta'}\,(\mathsf{LRK})}{\mathsf{K}_a\,\Gamma, \Gamma' \Rightarrow \mathsf{K}_a\,\varphi, \Delta'}\,(\mathsf{LK}^2)^*$$

□

3 Dynamic Epistemic Logic

We briefly summarise epistemic actions and dynamic epistemic logic following van Ditmarsch et al. [5]. Based on this we present our calculus $\mathbf{G4}_{P,A}[]$ and prove the admissibility of cut as well as its soundness and completeness.

An *epistemic action structure* $U = (Q, F, \mathit{pre})$ over (P, A) and some logical language $\mathcal{L}$ consists of a finite set of *action points* Q, an A-indexed family of epistemic *action accessibility relations* $F = (F_a \subseteq Q \times Q)_{a \in A}$, and a *precondition* function $\mathit{pre}\colon Q \to \mathcal{L}$. We assume that the accessibility relations are reflexive and transitive. For any agent a, $(q, q') \in F_a$ models that agent a cannot distinguish between occurrences of q and q'. For $q \in Q$, the epistemic formula $\mathit{pre}(q)$ determines a condition under which q can happen. An *epistemic action* $\mathfrak{u} = (U, q)$ over (P, A) and $\mathcal{L}$ is given by the epistemic action structure $U = (Q, F, \mathit{pre})$ and a designated point $q \in Q$.

The set $\Psi_{P,A}$ of *dynamic epistemic formulæ* over (P, A) is defined as $\bigcup_{n \in \mathbb{N}} \Psi^{(n)}_{P,A}$ where $\Psi^{(n)}_{P,A}$ are the dynamic epistemic formulæ of *depth* n; the set $\mathfrak{U}_{P,A}$ of *epistemic actions* over (P, A) is defined as $\bigcup_{n \in \mathbb{N}} \mathfrak{U}^{(n)}_{P,A}$ where $\mathfrak{U}^{(n)}_{P,A}$ are the epistemic actions of *depth* n. The families $(\Psi^{(n)}_{P,A})_{n \in \mathbb{N}}$ and $(\mathfrak{U}^{(n)}_{P,A})_{n \in \mathbb{N}}$ are mutually recursively defined as follows: $\Psi^{(0)}_{P,A}$ is just $\Phi_{P,A}$ and the dynamic epistemic formulæ $\Psi^{(n+1)}_{P,A}$ are defined by the following grammar:

$$\psi ::= \mathrm{false} \mid \psi_1 \supset \psi_2 \mid \mathsf{K}_a\,\psi \mid [\mathfrak{u}]\psi$$

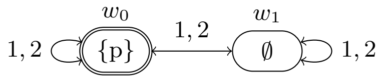
where $\mathfrak{u} \in \mathfrak{U}^{(n)}_{P,A}$; and $\mathfrak{U}^{(n)}_{P,A}$ comprises the epistemic actions over (P, A) and $\Psi^{(n)}_{P,A}$. The formula $[\mathfrak{u}]\psi$ is to be read as "the execution of the epistemic action $\mathfrak{u}$ in the current epistemic state leads to an epistemic state where the formula ψ holds". In the following ψ (and its adorned variants) always ranges over $\Psi_{P,A}$ and $\mathfrak{u}$ over $\mathfrak{U}_{P,A}$.

The *product update* of an epistemic structure $K = (W, E, L)$ over (P, A) and an epistemic action structure $U = (Q, F, \mathit{pre})$ over (P, A) is the epistemic structure $K \lhd U = (W', E', L')$ over (P, A) with

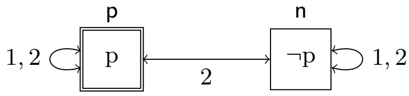
$$\begin{aligned} &W' = \{(w, q) \in W \times Q \mid K, w \models \mathit{pre}(q)\}\ , \\ &E'_a = \{((w, q), (w', q')) \in W' \times W' \mid (w, w') \in E_a,\ (q, q') \in F_a\} \text{ for all } a \in A, \\ &L'(w, q) = L(w) \text{ for all } (w, q) \in W'. \end{aligned}$$

The product update for epistemic structures is well-defined, since the relations E'_a are again reflexive and transitive. E'_a reflects that the uncertainty of an agent a in a world (w, q) is determined by the uncertainty of a about world w and its uncertainty about the occurrence of q. The *product update* of an epistemic state $\mathfrak{K}$ and an epistemic action $\mathfrak{u} = (U, q)$ over (P, A) is defined by the epistemic state $(K, w) \lhd (U, q) = (K \lhd U, (w, q))$, provided that $K, w \models \mathit{pre}(q)$. Note that all epistemic actions are deterministic.

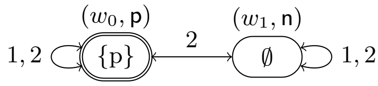
Example 1. *(see, e. g., [25]) Let $P = \{\mathrm{p}\}$ and $A = \{1, 2\}$. For the current epistemic state assume that neither agent 1 nor agent 2 know whether proposition p holds. This situation can be represented by $\mathfrak{K} = ((W, E, L), w_0)$ with $W = \{w_0, w_1\}$, $E_1 = W^2 = E_2$, and $L(w_0) = \{\mathrm{p}\}$, $L(w_1) = \emptyset$, as depicted below:*



(Both accessiblity relations are symmetric, as indicated by the arrows, but this is not required in **S4***). Now assume that only* 1 *reads a letter telling that* p*, such that* 1 *consequently knows* p*, but* 2 *does not. This reading is modelled by* $\mathfrak{r}\mathfrak{d} = ((Q, F, pre), \mathsf{p})$ *with* $Q = \{\mathsf{p}, \mathsf{n}\}$*,* $F_1 = \{(\mathsf{p}, \mathsf{p}), (\mathsf{n}, \mathsf{n})\}$*,* $F_2 = Q^2$*,* $pre(\mathsf{p}) = \mathrm{p}$*, and* $pre(\mathsf{n}) = \neg\mathrm{p}$*, graphically shown below:*



The epistemic state resulting from executing $\mathfrak{r}\mathfrak{d}$ *in* $\mathfrak{K}$*, depicted below, is* $\mathfrak{K} \lhd \mathfrak{r}\mathfrak{d} = ((W', E', L'), (w_0, \mathsf{p}))$ *with* $W' = \{(w_0, \mathsf{p}), (w_1, \mathsf{n})\}$*,* $E'_1 = \{((w_0, \mathsf{p}), (w_0, \mathsf{p})), ((w_1, \mathsf{n}), (w_1, \mathsf{n}))\}$*,* $E'_2 = {W'}^2$*,* $L'(w_0, \mathsf{p}) = \{\mathrm{p}\}$*, and* $L'(w_1, \mathsf{n}) = \emptyset$*:*



Indeed, in this epistemic state $\mathfrak{K} \lhd \mathfrak{r}\mathfrak{d}$ *agent* 1 *knows* p. □

The *syntactic composition* $U_1; U_2$ of two epistemic action structures $U_i = (Q_i, F_i, pre_i)$, $1 \leq i \leq 2$ is given by (Q, F, pre) with

$$Q = Q_1 \times Q_2 \ ,$$
$$F_a = \{((q_1, q_2), (q'_1, q'_2)) \mid (q_1, q'_1) \in F_{1,a},\ (q_2, q'_2) \in F_{2,a}\} \ ,$$
$$pre(q_1, q_2) = pre_1(q_1) \land [(U_1, q_1)]pre_2(q_2) \ .$$

The syntactic composition $\mathfrak{u}_1; \mathfrak{u}_2$ of two epistemic actions $\mathfrak{u}_i = (U_i, q_i)$, $1 \leq i \leq 2$, is given by $(U_1; U_2, (q_1, q_2))$. The syntactic composition of epistemic actions is associative up to isomorphism [5, Prop. 6.9], i. e., it holds for all $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3 \in \mathfrak{U}_{P,A}$ that

(A1) $\mathfrak{u}_1; (\mathfrak{u}_2; \mathfrak{u}_3) \cong (\mathfrak{u}_1; \mathfrak{u}_2); \mathfrak{u}_3$

In the following we will identify isomorphic epistemic actions.

For an epistemic action $\mathfrak{u} = ((Q, F, pre), q)$ we write $Q(\mathfrak{u})$ for Q, $F(\mathfrak{u})_a$ for $\{q' \mid (q, q') \in F_a\}$, ${}^{\cdot}\mathfrak{u}$ for $pre(q)$, $q(\mathfrak{u})$ for q, and $\mathfrak{u} \cdot q'$ for $((Q, F, pre), q')$ whenever $q' \in Q$. It holds for all $a \in A$, $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathfrak{U}_{P,A}$, and $q_i \in Q(\mathfrak{u}_i)$, $1 \leq i \leq 2$, that

(A2) $F(\mathfrak{u}_1; \mathfrak{u}_2)_a = F(\mathfrak{u}_1)_a \times F(\mathfrak{u}_2)_a$

(A3) $(\mathfrak{u}_1; \mathfrak{u}_2) \cdot (q_1, q_2) = (\mathfrak{u}_1 \cdot q_1); (\mathfrak{u}_2 \cdot q_2)$

(A4) ${}^{\cdot}(\mathfrak{u}_1; \mathfrak{u}_2) = {}^{\cdot}\mathfrak{u}_1 \land [\mathfrak{u}_1]{}^{\cdot}\mathfrak{u}_2$

The *satisfaction* of a dynamic epistemic formula ψ in an epistemic state $\mathfrak{K}$ over the same epistemic signature (P, A), written $\mathfrak{K} \models \psi$, extends the respective satisfaction of (pure) epistemic formulæ by

$$\mathfrak{K} \models [\mathfrak{u}]\psi \iff \mathfrak{K} \models {}^{\cdot}\mathfrak{u} \text{ implies } \mathfrak{K} \lhd \mathfrak{u} \models \psi \ .$$

The dynamic epistemic logic $\mathbf{DS4}_{P,A}$ consists of all dynamic epistemic formulæ $\psi \in \Psi_{P,A}$ such that $\mathfrak{K} \models \psi$ for all epistemic states $\mathfrak{K}$. This logic can be axiomatised in a Hilbert-calculus by the axioms and derivation rules for $\mathbf{S4}_{P,A}$, see Tab. 1, together with the *reduction axioms* in Tab. 5, where $\bigwedge$ abbreviates iterated conjunction.

(red$p$) $[\mathfrak{u}]p \leftrightarrow {}^{\cdot}\mathfrak{u} \supset p$	(redfalse) $[\mathfrak{u}]\mathrm{false} \leftrightarrow \neg\, {}^{\cdot}\mathfrak{u}$
(red$\supset$) $[\mathfrak{u}](\psi_1 \supset \psi_2) \leftrightarrow [\mathfrak{u}]\psi_1 \supset [\mathfrak{u}]\psi_2$	(redK) $[\mathfrak{u}]\mathsf{K}_a\, \psi \leftrightarrow {}^{\cdot}\mathfrak{u} \supset \bigwedge_{q \in F(\mathfrak{u})_a} \mathsf{K}_a[\mathfrak{u} \cdot q]\psi$
(red[]) $[\mathfrak{u}_1][\mathfrak{u}_2]\psi \leftrightarrow [\mathfrak{u}_1; \mathfrak{u}_2]\psi$	

Table 5: Reduction axioms for $\mathbf{DS4}_{P,A}$

Our Gentzen-style calculus $\mathbf{G4}_{P,A}[]$ for epistemic dynamic logic extends the epistemic rules in Tab. 2 with the action rules in Tab. 6 where now Γ and Δ always range over $\Psi_{P,A}$. Table 7 comprises some additional rules: On the one hand, the additional propositional connectives can be directly handled by corresponding derived rules; on the other hand, some admissible rules for handling actions are offered (see Lem. 9 and Lem. 10).

$$
(\mathrm{L}[]p)\ \frac{\Gamma \Rightarrow {}^{\cdot}\mathfrak{u}, \Delta \quad \Gamma, p \Rightarrow \Delta}{[\mathfrak{u}]p, \Gamma \Rightarrow \Delta}
\qquad
(\mathsf{R}[]p)\ \frac{{}^{\cdot}\mathfrak{u}, \Gamma \Rightarrow p, \Delta}{\Gamma \Rightarrow [\mathfrak{u}]p, \Delta}
$$

$$
(\mathrm{L}[]\mathrm{false})\ \frac{\Gamma \Rightarrow {}^{\cdot}\mathfrak{u}, \Delta}{[\mathfrak{u}]\mathrm{false}, \Gamma \Rightarrow \Delta}
\qquad
(\mathsf{R}[]\mathrm{false})\ \frac{{}^{\cdot}\mathfrak{u}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow [\mathfrak{u}]\mathrm{false}, \Delta}
$$

$$
(\mathrm{L}[]\supset)\ \frac{\Gamma \Rightarrow [\mathfrak{u}]\psi_1, \Delta \quad [\mathfrak{u}]\psi_2, \Gamma \Rightarrow \Delta}{[\mathfrak{u}](\psi_1 \supset \psi_2), \Gamma \Rightarrow \Delta}
\qquad
(\mathsf{R}[]\supset)\ \frac{[\mathfrak{u}]\psi_1, \Gamma \Rightarrow [\mathfrak{u}]\psi_2, \Delta}{\Gamma \Rightarrow [\mathfrak{u}](\psi_1 \supset \psi_2), \Delta}
$$

$$
(\mathrm{L}[]\mathsf{K})\ \frac{\Gamma \Rightarrow {}^{\cdot}\mathfrak{u}, \Delta \quad (\mathsf{K}_a[\mathfrak{u} \cdot q']\psi)_{q' \in F(\mathfrak{u})_a}, \Gamma \Rightarrow \Delta}{[\mathfrak{u}]\mathsf{K}_a\,\psi, \Gamma \Rightarrow \Delta}
\qquad
(\mathsf{R}[]\mathsf{K})\ \frac{({}^{\cdot}\mathfrak{u}, \Gamma \Rightarrow \mathsf{K}_a[\mathfrak{u} \cdot q']\psi, \Delta)_{q' \in F(\mathfrak{u})_a}}{\Gamma \Rightarrow [\mathfrak{u}]\mathsf{K}_a\,\psi, \Delta}
$$

$$
(\mathrm{L}[][])\ \frac{[\mathfrak{u}_1;\mathfrak{u}_2]\psi, \Gamma \Rightarrow \Delta}{[\mathfrak{u}_1][\mathfrak{u}_2]\psi, \Gamma \Rightarrow \Delta}
\qquad
(\mathsf{R}[][])\ \frac{\Gamma \Rightarrow [\mathfrak{u}_1;\mathfrak{u}_2]\psi, \Delta}{\Gamma \Rightarrow [\mathfrak{u}_1][\mathfrak{u}_2]\psi, \Delta}
$$

Table 6: Modal Gentzen system $\mathbf{G4}_{P,A}[]$ for dynamic epistemic logic

$$
(\mathrm{L}[]\neg)\ \frac{\Gamma \Rightarrow {}^{\cdot}\mathfrak{u}, \Delta \quad \Gamma \Rightarrow [\mathfrak{u}]\psi, \Delta}{[\mathfrak{u}]\neg\psi, \Gamma \Rightarrow \Delta}
\qquad
(\mathsf{R}[]\neg)\ \frac{{}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\psi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow [\mathfrak{u}]\neg\psi, \Delta}
$$

$$
(\mathrm{L}[]\wedge)\ \frac{[\mathfrak{u}]\psi_1, [\mathfrak{u}]\psi_2, \Gamma \Rightarrow \Delta}{[\mathfrak{u}](\psi_1 \wedge \psi_2), \Gamma \Rightarrow \Delta}
\qquad
(\mathsf{R}[]\wedge)\ \frac{\Gamma \Rightarrow [\mathfrak{u}]\psi_1, \Delta \quad \Gamma \Rightarrow [\mathfrak{u}]\psi_2, \Delta}{\Gamma \Rightarrow [\mathfrak{u}](\psi_1 \wedge \psi_2), \Delta}
$$

$$
(\mathrm{L}[]\vee)\ \frac{[\mathfrak{u}]\psi_1, \Gamma \Rightarrow \Delta \quad [\mathfrak{u}]\psi_2, \Gamma \Rightarrow \Delta}{[\mathfrak{u}](\psi_1 \vee \psi_2), \Gamma \Rightarrow \Delta}
\qquad
(\mathsf{R}[]\vee)\ \frac{\Gamma \Rightarrow [\mathfrak{u}]\psi_1, [\mathfrak{u}]\psi_2, \Delta}{\Gamma \Rightarrow [\mathfrak{u}](\psi_1 \vee \psi_2), \Delta}
$$

$$
(\mathsf{LK}[][])\ \frac{\mathsf{K}_a[\mathfrak{u}_1;\mathfrak{u}_2]\psi, \Gamma \Rightarrow \Delta}{\mathsf{K}_a[\mathfrak{u}_1][\mathfrak{u}_2]\psi, \Gamma \Rightarrow \Delta}
\qquad
(\mathsf{RK}[][])\ \frac{\Gamma \Rightarrow \mathsf{K}_a[\mathfrak{u}_1;\mathfrak{u}_2]\psi, \Delta}{\Gamma \Rightarrow \mathsf{K}_a[\mathfrak{u}_1][\mathfrak{u}_2]\psi, \Delta}
$$

$$
(\mathsf{LR}[])\ \frac{\Gamma \Rightarrow \Delta}{{}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Gamma, \Gamma' \Rightarrow [\mathfrak{u}]\Delta, \Delta'}
$$

Table 7: Additional rules for $\mathbf{G4}_{P,A}[]$

Example 2. *Consider the reading action of* $\mathfrak{rd}$ *as introduced in Ex. 1:*

$$
\dfrac{
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\dfrac{
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\dfrac{\vdots}{\neg\mathrm{p} \Rightarrow \neg\mathrm{p}} \quad \dfrac{\vdots}{\neg\mathrm{p}, \mathrm{p} \Rightarrow}
}{\neg\mathrm{p}, [\mathfrak{rd} \cdot \mathsf{n}]\mathrm{p} \Rightarrow}\ (\mathrm{L}[]p)
}{\Rightarrow [\mathfrak{rd} \cdot \mathsf{n}]\neg\mathrm{p}}\ (\mathsf{R}[]\neg)
}{\neg\mathrm{p} \Rightarrow [\mathfrak{rd} \cdot \mathsf{n}]\mathsf{K}_1\,\mathrm{p}, \mathsf{K}_1[\mathfrak{rd} \cdot \mathsf{n}]\neg\mathrm{p}}\ (\mathsf{RK})
}{\Rightarrow [\mathfrak{rd} \cdot \mathsf{n}]\mathsf{K}_1\,\mathrm{p}, [\mathfrak{rd} \cdot \mathsf{n}]\mathsf{K}_1\,\neg\mathrm{p}}\ (\mathsf{R}[]\mathsf{K})
}{\Rightarrow [\mathfrak{rd} \cdot \mathsf{n}](\mathsf{K}_1\,\mathrm{p} \vee \mathsf{K}_1\,\neg\mathrm{p})}\ (\mathsf{R}[]\vee)
}{\mathrm{p} \Rightarrow \mathsf{K}_2[\mathfrak{rd} \cdot \mathsf{n}](\mathsf{K}_1\,\mathrm{p} \vee \mathsf{K}_1\,\neg\mathrm{p})}\ (\mathsf{RK})
\quad
\dfrac{\vdots}{\mathrm{p} \Rightarrow \mathsf{K}_2[\mathfrak{rd} \cdot \mathsf{p}](\mathsf{K}_1\,\mathrm{p} \vee \mathsf{K}_1\,\neg\mathrm{p})}
}{\Rightarrow [\mathfrak{rd}]\mathsf{K}_2(\mathsf{K}_1\,\mathrm{p} \vee \mathsf{K}_1\,\neg\mathrm{p})}\ (\mathsf{R}[]\mathsf{K})
\qquad \square
$$

The *rank* of a formula ψ and an action $\mathfrak{u}$ is inductively defined as follows (see [5, Def. 7.38]):

$$
\begin{aligned}
&rk(\mathrm{false}) = 1 \\
&rk(p) = 1 \\
&rk(\psi_1 \supset \psi_2) = 1 + \max\{rk(\psi_1), rk(\psi_2)\} \\
&rk(\mathsf{K}_a\,\psi) = 1 + rk(\psi) \\
&rk([\mathfrak{u}]\psi) = (4 + rk(\mathfrak{u})) \cdot rk(\psi) \\
&rk(\mathfrak{u}) = \max\{rk({}^{\cdot}(\mathfrak{u} \cdot q)) \mid q \in Q(\mathfrak{u})\}
\end{aligned}
$$

It holds that $rk([\mathfrak{u}]\mathsf{K}_a\,\psi) > rk([\mathfrak{u}]\psi)$, $rk([\mathfrak{u}]\psi) > rk(\dot{}\mathfrak{u})$, $rk([\mathfrak{u}]\psi) > rk(\psi)$, $rk([\mathfrak{u}]\mathsf{K}_a\,\psi) > rk(\mathsf{K}_a[\mathfrak{u}\cdot q]\psi)$ for all $q \in Q(\mathfrak{u})$, and $rk([\mathfrak{u}_1][\mathfrak{u}_2]\psi) > rk([\mathfrak{u}_1;\mathfrak{u}_2]\psi)$.

The following lemmata hold for all $\psi \in \Psi_{P,A}$, $\mathfrak{u},\mathfrak{u}_1,\mathfrak{u}_2 \in \mathfrak{U}_{P,A}$, $a \in A$, and $\Psi_{P,A}$-multisets Γ and Δ. We first show that Lem. 1 generalising the axiom rule $(p\mathrm{A})$ to arbitrary formulæ carries over from $\mathbf{G4}_{P,A}$.

Lemma 5. $\vdash_{\mathbf{G4}_{P,A}[]} \psi,\Gamma \Rightarrow \Delta,\psi$.

Proof. We proceed by induction on the rank of ψ. For $\psi \in \Phi_{P,A}$ the claim already holds in $\mathbf{G4}_{P,A}$ by Lem. 1. We only consider $\psi = [\mathfrak{u}]p$ and $\psi = [\mathfrak{u}]\mathsf{K}_a\,\psi'$, the remaining cases are analogous.
Case $\psi = [\mathfrak{u}]p$: We have

$$\dfrac{\dfrac{\dot{}\mathfrak{u},\Gamma \Rightarrow \overset{\vdots\,\text{I. H.}}{\dot{}\mathfrak{u},p,\Delta} \qquad \dot{}\mathfrak{u},p,\Gamma \overset{\vdots\,\text{I. H.}}{\Rightarrow} p,\Delta}{\dot{}\mathfrak{u},[\mathfrak{u}]p,\Gamma \Rightarrow p,\Delta}\,(\mathsf{L}[]p)}{[\mathfrak{u}]p,\Gamma \Rightarrow [\mathfrak{u}]p,\Delta}\,(\mathsf{R}[]p)$$

Case $\psi = [\mathfrak{u}]\mathsf{K}_a\,\psi'$: We have

$$\dfrac{\dfrac{(\dot{}\mathfrak{u},\Gamma \Rightarrow \overset{\vdots\,\text{I. H.}}{\dot{}\mathfrak{u},[\mathfrak{u}\cdot q']\mathsf{K}_a\,\psi',\Delta})_{q'\in F(\mathfrak{u})_a} \qquad (\dot{}\mathfrak{u},([\mathfrak{u}\cdot q'']\mathsf{K}_a\,\psi')_{q''\in F(\mathfrak{u})},\overset{\vdots\,\text{I. H.}}{\Gamma} \Rightarrow [\mathfrak{u}\cdot q']\mathsf{K}_a\,\psi',\Delta)_{q'\in F(\mathfrak{u})_a}}{(\dot{}\mathfrak{u},[\mathfrak{u}]\mathsf{K}_a\,\psi',\Gamma \Rightarrow [\mathfrak{u}\cdot q']\mathsf{K}_a\,\psi',\Delta)_{q'\in F(\mathfrak{u})_a}}\,(\mathsf{L}[]\mathsf{K})}{[\mathfrak{u}]\mathsf{K}_a\,\psi',\Gamma \Rightarrow [\mathfrak{u}]\mathsf{K}_a\,\psi',\Delta}\,(\mathsf{R}[]\mathsf{K})$$

$\square$

Also, Lem. 2 showing the admissibility of (Weak) and (Contr) carries over from $\mathbf{G4}_{P,A}$.

Lemma 6. (Weak) *and* (Contr) *are height-preservingly admissible for* $\mathbf{G4}_{P,A}[]$.

Proof. By induction on the height of the derivation. $\square$

Lemma 7. *If* $\vdash_{\mathbf{G4}_{P,A}[]} \dot{}\mathfrak{u},\Gamma \Rightarrow [\mathfrak{u}]\psi,\Delta$, *then* $\vdash_{\mathbf{G4}_{P,A}[]} \Gamma \Rightarrow [\mathfrak{u}]\psi,\Delta$.

Proof. We proceed by induction over the derivation of $\vdash_{\mathbf{G4}_{P,A}[]} \dot{}\mathfrak{u},\Gamma \Rightarrow [\mathfrak{u}]\psi,\Delta$ and consider the last rule applied. If therein $[\mathfrak{u}]\psi$ is principal and $\dot{}\mathfrak{u}$ is added in the antecedent — as in $(\mathsf{R}[]p)$, $(\mathsf{R}[]\mathrm{false})$, and $(\mathsf{R}[]\mathsf{K})$ —, then (Contr) is applied. If, e. g., $(\mathsf{R}[]p)$ is the last rule, then $\psi = p$ and $\vdash_{\mathbf{G4}_{P,A}[]} \dot{}\mathfrak{u},\dot{}\mathfrak{u},\Gamma \Rightarrow p,\Delta$, and

$$\dfrac{\dfrac{\dot{}\mathfrak{u},\dot{}\mathfrak{u},\Gamma \Rightarrow p,\Delta}{\dot{}\mathfrak{u},\Gamma \Rightarrow p,\Delta}\,(\mathrm{Contr})}{\Gamma \Rightarrow [\mathfrak{u}]p,\Delta}\,(\mathsf{R}[]p)$$

If $[\mathfrak{u}]\psi$ is principal, but $\dot{}\mathfrak{u}$ is not added in the antecedent — as in $(\mathsf{R}[]\supset)$ and $(\mathsf{R}[][])$ —, then the claim follows directly from the induction hypothesis.

If $[\mathfrak{u}]\psi$ is not principal and the rule for $[\mathfrak{u}]\psi$ in the succedent adds $\dot{}\mathfrak{u}$ to the antecedent, we first (from bottom to top) duplicate $[\mathfrak{u}]\psi$ in the succedent by (Contr), then apply the "box"-rule matching ψ for adding $\dot{}\mathfrak{u}$, and finally apply (Weak):

$$\dfrac{\dfrac{\ldots \qquad \dfrac{\dot{}\mathfrak{u},\Gamma \Rightarrow [\mathfrak{u}]\psi,\Delta}{\dot{}\mathfrak{u},\Gamma,\Gamma' \Rightarrow [\mathfrak{u}]\psi,\Delta',\Delta}\,(\mathrm{Weak})}{\Gamma \Rightarrow [\mathfrak{u}]\psi,[\mathfrak{u}]\psi,\Delta}\,\mathsf{R}[]\ldots}{\Gamma \Rightarrow [\mathfrak{u}]\psi,\Delta}\,(\mathrm{Contr})$$

If $[\mathfrak{u}]\psi$ is not principal and $\dot{}\mathfrak{u}$ is not added to the antecedent by the "box"-rule matching ψ, then the claim follows directly from the induction hypothesis. $\square$

Lemma 8. *All of the "box" rules in* $\mathbf{G4}_{P,A}[]$ *are invertible, i. e.: if* $\vdash_{\mathbf{G4}_{P,A}[]} [\mathfrak{u}]p,\Gamma \Rightarrow \Delta$, *then* $\vdash_{\mathbf{G4}_{P,A}[]} \Gamma \Rightarrow \dot{}\mathfrak{u},\Delta$ *and* $\vdash_{\mathbf{G4}_{P,A}[]} \Gamma,p \Rightarrow \Delta$, *&c.*

Proof. Only a single rule applies to each possible form of $[\mathfrak{u}]\psi$ in the antecedent and the succedent. $\square$

We append $^{-1}$ to a rule name when applying it invertedly. The rules $(\mathsf{LK}[][])$ and $(\mathsf{RK}[][])$ for treating sequential composition of epistemic actions and repeated boxes equally are both admissible and invertible:

Lemma 9. *(a)* $\vdash_{\mathbf{G4}_{P,A}[]} \mathsf{K}_a[\mathfrak{u}_1;\mathfrak{u}_2]\psi, \Gamma \Rightarrow \Delta$ *if, and only if,* $\vdash_{\mathbf{G4}_{P,A}[]} \mathsf{K}_a[\mathfrak{u}_1][\mathfrak{u}_2]\psi, \Gamma \Rightarrow \Delta$.
(b) $\vdash_{\mathbf{G4}_{P,A}[]} \Gamma \Rightarrow \mathsf{K}_a[\mathfrak{u}_1;\mathfrak{u}_2]\psi, \Delta$ *if, and only if,* $\vdash_{\mathbf{G4}_{P,A}[]} \Gamma \Rightarrow \mathsf{K}_a[\mathfrak{u}_1][\mathfrak{u}_2]\psi, \Delta$.

Proof. (a) The only applicable rule with $\mathsf{K}_a[\mathfrak{u}_1;\mathfrak{u}_2]\psi$ and $\mathsf{K}_a[\mathfrak{u}_1][\mathfrak{u}_2]\psi$ principal is (LT) where the claim follows immediately by (L[][]) or $(\mathrm{L}[][])^{-1}$.

(b) Only (RK) shows $\mathsf{K}_a[\mathfrak{u}_1;\mathfrak{u}_2]\psi$ or $\mathsf{K}_a[\mathfrak{u}_1][\mathfrak{u}_2]\psi$ principally. If the last rule for obtaining $\vdash_{\mathbf{G4}_{P,A}[]} \Gamma \Rightarrow \mathsf{K}_a[\mathfrak{u}_1;\mathfrak{u}_2]\psi, \Delta$ has been (RK), then $\Gamma = \mathsf{K}_a\,\Gamma', \Gamma''$ for some Γ', Γ'', and $\vdash_{\mathbf{G4}_{P,A}[]} \mathsf{K}_a\,\Gamma' \Rightarrow [\mathfrak{u}_1;\mathfrak{u}_2]\psi$. We thus have

$$\dfrac{\dfrac{\mathsf{K}_a\,\Gamma' \Rightarrow [\mathfrak{u}_1;\mathfrak{u}_2]\psi}{\mathsf{K}_a\,\Gamma' \Rightarrow [\mathfrak{u}_1][\mathfrak{u}_2]\psi}\;(\mathrm{R}[][])}{\mathsf{K}_a\,\Gamma', \Gamma'' \Rightarrow \mathsf{K}_a[\mathfrak{u}_1][\mathfrak{u}_2]\psi, \Delta}\;(\mathsf{RK})$$

The reverse direction uses $(\mathrm{R}[][])^{-1}$. □

We show that the rule (LR[]) is admissible. The rule always assumes the precondition of the contextual epistemic action to hold; without this precondition, the rule would not apply to an empty succedent (see Lem. 7): The sequent false $\Rightarrow$ is derivable, but $[\mathfrak{ff}]$false $\Rightarrow$ with ${}^{\cdot}\mathfrak{ff} = $ false must not be.

Lemma 10. *If* $\vdash_{\mathbf{G4}_{P,A}[]} \Gamma \Rightarrow \Delta$*, then* $\vdash_{\mathbf{G4}_{P,A}[]} {}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Gamma, \Gamma' \Rightarrow [\mathfrak{u}]\Delta, \Delta'$ *for all* $\mathfrak{u}$, Γ'*, and* Δ'.

Proof. We proceed by induction over the derivation of $\vdash_{\mathbf{G4}_{P,A}[]} \Gamma \Rightarrow \Delta$ and consider the last rule applied.
Case (pA): Then $\Gamma = p, \Gamma'$ and $\Delta = p, \Delta'$ for some p, Γ', Δ'. We have

$$\dfrac{\dfrac{\overset{\vdots\,\text{Lem. 5}}{{}^{\cdot}\mathfrak{u}, {}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow {}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Delta', \Delta''} \qquad \dfrac{}{{}^{\cdot}\mathfrak{u}, {}^{\cdot}\mathfrak{u}, p, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow p, [\mathfrak{u}]\Delta', \Delta''}\;(p\mathrm{A})}{{}^{\cdot}\mathfrak{u}, {}^{\cdot}\mathfrak{u}, [\mathfrak{u}]p, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow p, [\mathfrak{u}]\Delta', \Delta''}\;(\mathrm{L}[]p)}{{}^{\cdot}\mathfrak{u}, [\mathfrak{u}]p, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]p, [\mathfrak{u}]\Delta', \Delta''}\;(\mathrm{R}[]p)$$

Case (Lfalse): Then $\Gamma = \text{false}, \Gamma'$ for some Γ'. We have

$$\dfrac{\overset{\vdots\,\text{Lem. 5}}{{}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow {}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Delta', \Delta''}}{{}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\text{false}, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'}\;(\mathrm{L}[]\text{false})$$

Case (L$\supset$): Then $\Gamma = \psi_1 \supset \psi_2, \Gamma'$ for some ψ_1, ψ_2, Γ', and $\vdash_{\mathbf{G4}_{P,A}[]} \Gamma' \Rightarrow \psi_1, \Delta$ as well as $\vdash_{\mathbf{G4}_{P,A}[]} \psi_2, \Gamma' \Rightarrow \Delta$. We have

$$\dfrac{\overset{\vdots\,\text{I. H.}}{{}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\psi_1, [\mathfrak{u}]\Delta, \Delta'} \qquad \overset{\vdots\,\text{I. H.}}{{}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\psi_2, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'}}{{}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\psi_1 \supset \psi_2, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'}\;(\mathrm{L}[]\supset)$$

Case (R$\supset$): Then $\Delta = \psi_1 \supset \psi_2, \Delta'$, for some ψ_1, ψ_2, Δ', and $\vdash_{\mathbf{G4}_{P,A}[]} \psi_1, \Gamma \Rightarrow \psi_2, \Delta'$. We have

$$\dfrac{\overset{\vdots\,\text{I. H.}}{{}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\psi_1, [\mathfrak{u}]\Gamma, \Gamma' \Rightarrow [\mathfrak{u}]\psi_2, [\mathfrak{u}]\Delta', \Delta''}}{{}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Gamma, \Gamma' \Rightarrow [\mathfrak{u}]\psi_1 \supset \psi_2, [\mathfrak{u}]\Delta', \Delta''}\;(\mathrm{R}[]\supset)$$

Case (LT): Then $\Gamma = \mathsf{K}_a\,\psi, \Gamma'$ for some a, ψ, Γ', and $\vdash_{\mathbf{G4}_{P,A}[]} \psi, \Gamma' \Rightarrow \Delta$. We have

$$\dfrac{\overset{\vdots\,\text{Lem. 5}}{{}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow {}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Delta, \Delta'} \qquad \dfrac{\dfrac{\overset{\vdots\,\text{I. H.}}{{}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\psi, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'}}{{}^{\cdot}\mathfrak{u}, \mathsf{K}_a[\mathfrak{u}]\psi, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'}\;(\mathrm{LT})}{{}^{\cdot}\mathfrak{u}, (\mathsf{K}_a[\mathfrak{u}\cdot q']\psi)_{q' \in F(\mathfrak{u})_a}, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'}\;(\mathrm{Weak})}{{}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\mathsf{K}_a\,\psi, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'}\;(\mathrm{L}[]\mathsf{K})$$

where the step (Weak) is possible since $q(\mathfrak{u}) \in F(\mathfrak{u})_a$ by the reflexivity of $F(\mathfrak{u})$ and $\mathfrak{u} \cdot q(\mathfrak{u}) = \mathfrak{u}$.

Case (RK): Then $\Gamma = \mathsf{K}_a\,\Gamma', \Gamma''$ and $\Delta = \mathsf{K}_a\,\psi, \Delta'$ for some $a, \Gamma', \Gamma'', \psi, \Delta'$, and $\vdash_{\mathbf{G4}_{P,A[]}} \mathsf{K}_a\,\Gamma' \Rightarrow \psi$. We have

$$
\dfrac{
\begin{array}{c}\vdots\,\text{Lem. 5}\\ {}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Gamma'', \Gamma''' \Rightarrow\\ {}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\mathsf{K}_a\,\psi, [\mathfrak{u}]\Delta', \Delta''\end{array}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\begin{array}{c}\vdots\,\text{I. H.}\\ {}^{\cdot}(\mathfrak{u}\cdot q'), \big([\mathfrak{u}\cdot q']\mathsf{K}_a\,\Gamma' \Rightarrow [\mathfrak{u}\cdot q']\psi\big)_{q'\in F(\mathfrak{u})_a}\end{array}
}{
\big([\mathfrak{u}\cdot q']\mathsf{K}_a\,\Gamma' \Rightarrow [\mathfrak{u}\cdot q']\psi\big)_{q'\in F(\mathfrak{u})_a}
}\ \text{Lem. 7}
}{
\big((\mathsf{K}_a[(\mathfrak{u}\cdot q')\cdot q'']\Gamma')_{q'\in F(\mathfrak{u})_a, q''\in F(\mathfrak{u}\cdot q')_a} \Rightarrow [\mathfrak{u}\cdot q']\psi\big)_{q'\in F(\mathfrak{u})_a}
}\ (\mathsf{L}[]\mathsf{K})^{-1}
}{
\big((\mathsf{K}_a[\mathfrak{u}\cdot q'']\Gamma')_{q''\in F(\mathfrak{u})_a} \Rightarrow [\mathfrak{u}\cdot q']\psi\big)_{q'\in F(\mathfrak{u})_a}
}\ (\text{Contr})
}{
\big({}^{\cdot}\mathfrak{u}, (\mathsf{K}_a[\mathfrak{u}\cdot q']\Gamma')_{q''\in F(\mathfrak{u})_a}, [\mathfrak{u}]\Gamma'', \Gamma''' \Rightarrow \mathsf{K}_a[\mathfrak{u}\cdot q']\psi, [\mathfrak{u}]\Delta', \Delta''\big)_{q'\in F(\mathfrak{u})_a}
}\ (\mathsf{RK})
}{
{}^{\cdot}\mathfrak{u}, (\mathsf{K}_a[\mathfrak{u}\cdot q'']\Gamma')_{q''\in F(\mathfrak{u})_a}, [\mathfrak{u}]\Gamma'', \Gamma''' \Rightarrow [\mathfrak{u}]\mathsf{K}_a\,\psi, [\mathfrak{u}]\Delta', \Delta''
}\ (\mathsf{R}[]\mathsf{K})
}{
{}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\mathsf{K}_a\,\Gamma', [\mathfrak{u}]\Gamma'', \Gamma''' \Rightarrow [\mathfrak{u}]\mathsf{K}_a\,\psi, [\mathfrak{u}]\Delta', \Delta''
}\ (\mathsf{L}[]\mathsf{K})
$$

where $\big((\mathfrak{u}\cdot q')\cdot q''\big)_{q'\in F(\mathfrak{u})_a, q''\in F(\mathfrak{u}\cdot q')_a} = (\mathfrak{u}\cdot q')_{q'\in F(\mathfrak{u})_a}$ up to contraction by transitivity and reflexivity.

Case (L$[]p$): Then $\Gamma = [\mathfrak{u}']p, \Gamma'$ for some $\mathfrak{u}', p, \Gamma'$, and $\vdash_{\mathbf{G4}_{P,A[]}} \Gamma' \Rightarrow {}^{\cdot}\mathfrak{u}', \Delta$ as well as $\vdash_{\mathbf{G4}_{P,A[]}} \Gamma', p \Rightarrow \Delta$.

$$
\dfrac{
\dfrac{
\begin{array}{c}\vdots\,*_0\\ {}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow {}^{\cdot}(\mathfrak{u};\mathfrak{u}'), [\mathfrak{u}]\Delta, \Delta'\end{array}
\qquad
\dfrac{
\begin{array}{c}\vdots\,\text{I. H.}\\ {}^{\cdot}\mathfrak{u}, [\mathfrak{u}]p, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'\end{array}
}{
{}^{\cdot}\mathfrak{u}, p, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'
}\ (\mathrm{L}[]p)^{-1}
}{
{}^{\cdot}\mathfrak{u}, [\mathfrak{u};\mathfrak{u}']p, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'
}\ (\mathrm{L}[]p)
}{
{}^{\cdot}\mathfrak{u}, [\mathfrak{u}][\mathfrak{u}']p, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'
}\ (\mathrm{L}[][])
$$

where the derivation $*_0$ is

$$
\dfrac{
\begin{array}{c}\vdots\,\text{Lem. 5}\\ {}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow {}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Delta, \Delta'\end{array}
\quad
\begin{array}{c}\vdots\,\text{I. H.}\\ {}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]{}^{\cdot}\mathfrak{u}', [\mathfrak{u}]\Delta, \Delta'\end{array}
}{
{}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow {}^{\cdot}(\mathfrak{u};\mathfrak{u}'), [\mathfrak{u}]\Delta, \Delta'
}\ (\mathrm{R}\wedge)
$$

Case (R$[]p$): Then $\Delta = [\mathfrak{u}']p, \Delta'$ for some $\mathfrak{u}', p, \Delta'$, and $\vdash_{\mathbf{G4}_{P,A[]}} {}^{\cdot}\mathfrak{u}', \Gamma \Rightarrow p, \Delta'$.

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\begin{array}{c}\vdots\,\text{I. H.}\\ {}^{\cdot}\mathfrak{u}, [\mathfrak{u}]{}^{\cdot}\mathfrak{u}', [\mathfrak{u}]\Gamma, \Gamma' \Rightarrow [\mathfrak{u}]p, [\mathfrak{u}]\Delta', \Delta''\end{array}
}{
{}^{\cdot}\mathfrak{u}, {}^{\cdot}\mathfrak{u}, [\mathfrak{u}]{}^{\cdot}\mathfrak{u}', [\mathfrak{u}]\Gamma, \Gamma' \Rightarrow p, [\mathfrak{u}]\Delta', \Delta''
}\ (\mathrm{R}[]p)^{-1}
}{
{}^{\cdot}\mathfrak{u}, {}^{\cdot}(\mathfrak{u};\mathfrak{u}'), [\mathfrak{u}]\Gamma, \Gamma' \Rightarrow p, [\mathfrak{u}]\Delta', \Delta''
}\ (\mathrm{L}\wedge)
}{
{}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Gamma, \Gamma' \Rightarrow [\mathfrak{u};\mathfrak{u}']p, [\mathfrak{u}]\Delta', \Delta''
}\ (\mathrm{R}[]p)
}{
{}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Gamma, \Gamma' \Rightarrow [\mathfrak{u}][\mathfrak{u}']p, [\mathfrak{u}]\Delta', \Delta''
}\ (\mathrm{R}[][])
$$

Case (L[]false): Then $\Gamma = [\mathfrak{u}']\mathrm{false}, \Gamma'$ for some $\mathfrak{u}', \Gamma'$, and $\vdash_{\mathbf{G4}_{P,A[]}} \Gamma' \Rightarrow {}^{\cdot}\mathfrak{u}', \Delta$. We have

$$
\dfrac{
\dfrac{
\begin{array}{c}\vdots\,*_0\\ {}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow {}^{\cdot}(\mathfrak{u};\mathfrak{u}'), [\mathfrak{u}]\Delta, \Delta'\end{array}
}{
{}^{\cdot}\mathfrak{u}, [\mathfrak{u};\mathfrak{u}']\mathrm{false}, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'
}\ (\mathrm{L}[]\mathrm{false})
}{
{}^{\cdot}\mathfrak{u}, [\mathfrak{u}][\mathfrak{u}']\mathrm{false}, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'
}\ (\mathrm{L}[][])
$$

where the derivation $*_0$ is as in (L$[]p$).

Case (R[]false): Analogous to (L[]false).

Case (L$[]\supset$): Then $\Gamma = [\mathfrak{u}']\psi_1 \supset \psi_2, \Gamma'$ for some $\mathfrak{u}', \psi_1, \psi_2, \Gamma'$, and $\vdash_{\mathbf{G4}_{P,A[]}} \Gamma' \Rightarrow [\mathfrak{u}']\psi_1, \Delta$ as well as $\vdash_{\mathbf{G4}_{P,A[]}} [\mathfrak{u}']\psi_2, \Gamma' \Rightarrow \Delta$. We have

$$
\dfrac{
\dfrac{
\dfrac{
\begin{array}{c}\vdots\,\text{I. H.}\\ {}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}][\mathfrak{u}']\psi_1, [\mathfrak{u}]\Delta, \Delta'\end{array}
}{
{}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u};\mathfrak{u}']\psi_1, [\mathfrak{u}]\Delta, \Delta'
}\ (\mathrm{R}[][])^{-1}
\quad
\dfrac{
\begin{array}{c}\vdots\,\text{I. H.}\\ {}^{\cdot}\mathfrak{u}, [\mathfrak{u}][\mathfrak{u}']\psi_2, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'\end{array}
}{
{}^{\cdot}\mathfrak{u}, [\mathfrak{u};\mathfrak{u}']\psi_2, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'
}\ (\mathrm{L}[][])^{-1}
}{
{}^{\cdot}\mathfrak{u}, [\mathfrak{u};\mathfrak{u}'](\psi_1 \supset \psi_2), [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'
}\ (\mathrm{L}[]\supset)
}{
{}^{\cdot}\mathfrak{u}, [\mathfrak{u}][\mathfrak{u}'](\psi_1 \supset \psi_2), [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'
}\ (\mathrm{L}[][])
$$

Case (R[]⊃): Analogous to (L[]⊃).

Case (L[]K): Then $\Gamma = [\mathfrak{u}']\mathsf{K}_a\,\psi, \Gamma'$ for some $\mathfrak{u}', a, \psi, \Gamma'$, and $\vdash_{\mathbf{G4}_{P,A}[]} \Gamma' \Rightarrow {}^{\cdot}\mathfrak{u}', \Delta$ as well as $\vdash_{\mathbf{G4}_{P,A}[]} (\mathsf{K}_a[\mathfrak{u}' \cdot q']\psi)_{q' \in F(\mathfrak{u}')_a}, \Gamma' \Rightarrow \Delta$. We have

$$\dfrac{\dfrac{\overset{\vdots *_0}{{}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow {}^{\cdot}(\mathfrak{u};\mathfrak{u}'), [\mathfrak{u}]\Delta, \Delta'} \quad \overset{\vdots *_1}{{}^{\cdot}\mathfrak{u}, (\mathsf{K}_a[(\mathfrak{u};\mathfrak{u}') \cdot q'']\psi)_{q'' \in F(\mathfrak{u};\mathfrak{u}')_a}, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'}}{{}^{\cdot}\mathfrak{u}, [\mathfrak{u};\mathfrak{u}']\mathsf{K}_a\,\psi, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'}\ (\mathrm{L}[]\mathsf{K})}{{}^{\cdot}\mathfrak{u}, [\mathfrak{u}][\mathfrak{u}']\mathsf{K}_a\,\psi, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'}\ (\mathrm{L}[][])$$

where the derivation $*_0$ is as in the case of (L[]p) and and the derivation $*_1$ is

$$\dfrac{\dfrac{\overset{\vdots\,\mathrm{I.\,H.}}{{}^{\cdot}\mathfrak{u}, ([\mathfrak{u}]\mathsf{K}_a[\mathfrak{u}' \cdot q']\psi)_{q' \in F(\mathfrak{u}')_a}, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'}}{{}^{\cdot}\mathfrak{u}, (\mathsf{K}_a[\mathfrak{u} \cdot q][\mathfrak{u}' \cdot q']\psi)_{q \in F(\mathfrak{u})_a, q' \in F(\mathfrak{u}')_a}, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'}\ (\mathrm{L}[]\mathsf{K})^{-1}}{{}^{\cdot}\mathfrak{u}, (\mathsf{K}_a[(\mathfrak{u};\mathfrak{u}') \cdot q'']\psi)_{q'' \in F(\mathfrak{u};\mathfrak{u}')_a}, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'}\ (\mathrm{L}\mathsf{K}[][])^{-1}$$

Case (R[]K): Analogous to (L[]K).

Case (L[][]): Then $\Gamma = [\mathfrak{u}_1][\mathfrak{u}_2]\mathsf{K}_a\,\psi, \Gamma'$ for some $\mathfrak{u}_1, \mathfrak{u}_2, a, \psi$, and $\vdash_{\mathbf{G4}_{P,A}[]} [\mathfrak{u}_1;\mathfrak{u}_2]\psi, \Gamma' \Rightarrow \Delta$. We have

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\vdots\,\mathrm{I.\,H.}}{{}^{\cdot}\mathfrak{u}, [\mathfrak{u}][\mathfrak{u}_1;\mathfrak{u}_2]\psi, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'}}{{}^{\cdot}\mathfrak{u}, [\mathfrak{u};(\mathfrak{u}_1;\mathfrak{u}_2)]\psi, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'}\ (\mathrm{L}[][])^{-1}}{{}^{\cdot}\mathfrak{u}, [(\mathfrak{u};\mathfrak{u}_1);\mathfrak{u}_2]\psi, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'}\ (\mathrm{A1})}{{}^{\cdot}\mathfrak{u}, [\mathfrak{u};\mathfrak{u}_1][\mathfrak{u}_2]\psi, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'}\ (\mathrm{L}[][])}{{}^{\cdot}\mathfrak{u}, [\mathfrak{u}][\mathfrak{u}_1][\mathfrak{u}_2]\psi, [\mathfrak{u}]\Gamma', \Gamma'' \Rightarrow [\mathfrak{u}]\Delta, \Delta'}\ (\mathrm{L}[][])$$

Case (R[][]): Analogous to (L[][]). □

We finally show that (Cut) is admissible for $\mathbf{G4}_{P,A}[]$. First, we prove the admissibility of the cut rule for independent contexts Γ_1, Δ_1 and Γ_2, Δ_2:

Lemma 11. *For any $\psi \in \Psi_{P,A}$ and any multisets $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$ of $\Psi_{P,A}$-formulæ it holds that $\vdash_{\mathbf{G4}_{P,A}[]} \Gamma_1 \Rightarrow \Delta_1, \psi$ and $\vdash_{\mathbf{G4}_{P,A}[]} \psi, \Gamma_2 \Rightarrow \Delta_2$ implies $\vdash_{\mathbf{G4}_{P,A}[]} \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$.*

Proof. We proceed by a double induction over the rank $rk(\psi)$ of ψ and the height of a deduction.

Case 1: At least one of the sequents of the hypothesis of the claim is an axiom. A proof of the form

$$\dfrac{\dfrac{}{p, \Gamma_1 \Rightarrow \Delta_1, p}\ (p\mathrm{A}) \quad \overset{\vdots\,h}{p, \Gamma_2 \Rightarrow \Delta_2}}{p, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\mathrm{Cut})$$

is transformed into

$$\dfrac{\overset{\vdots\,h}{p, \Gamma_2 \Rightarrow \Delta_2}}{p, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\mathrm{Weak})$$

Case 2: The cut-formula is a side formula ψ. We only give some illustrative cases, the transformations for all the other rules is analogous.

Case 2–(RK): Then

$$\dfrac{\dfrac{\overset{\vdots\,h_1}{\mathsf{K}_a\,\Gamma' \Rightarrow \psi'}}{\mathsf{K}_a\,\Gamma', \Gamma_1 \Rightarrow \mathsf{K}_a\,\psi', \Delta_1, \psi}\ (\mathrm{R}\mathsf{K}) \quad \overset{\vdots\,h_2}{\psi, \Gamma_2 \Rightarrow \Delta_2}}{\mathsf{K}_a\,\Gamma', \Gamma_1, \Gamma_2 \Rightarrow \mathsf{K}_a\,\psi', \Delta_1, \Delta_2}\ (\mathrm{Cut})$$

is transformed into

$$\dfrac{\overset{\vdots\, h_1}{\mathsf{K}_a\,\Gamma' \Rightarrow \psi'}}{\mathsf{K}_a\,\Gamma', \Gamma_1, \Gamma_2 \Rightarrow \mathsf{K}_a\,\psi', \Delta_1, \Delta_2}\;(\mathsf{RK})$$

Case 2–(R$\supset$): Then

$$\dfrac{\dfrac{\overset{\vdots\, h_1}{\psi_1', \Gamma_1 \Rightarrow \psi_2', \Delta_1, \psi}}{\Gamma_1 \Rightarrow \psi_1' \supset \psi_2', \Delta_1, \psi}\;(\mathsf{R}\supset) \qquad \overset{\vdots\, h_2}{\psi, \Gamma_2 \Rightarrow \Delta_2}}{\Gamma_1, \Gamma_2 \Rightarrow \psi_1' \supset \psi_2', \Delta_1, \Delta_2}\;(\mathrm{Cut})$$

is transformed into

$$\dfrac{\dfrac{\overset{\vdots\, h_1}{\psi_1', \Gamma_1 \Rightarrow \psi_2', \Delta_1, \psi} \qquad \overset{\vdots\, h_2}{\psi, \Gamma_2 \Rightarrow \Delta_2}}{\psi_1', \Gamma_1, \Gamma_2 \Rightarrow \psi_2', \Delta_1, \Delta_2}\;(\mathrm{Cut})}{\Gamma_1, \Gamma_2 \Rightarrow \psi_1' \supset \psi_2', \Delta_1, \Delta_2}\;(\mathsf{R}\supset)$$

Case 2–(L[]K): Then

$$\dfrac{\dfrac{\overset{\vdots\, h_{11}}{\Gamma_1 \Rightarrow {}^{\cdot}\mathfrak{u}, \Delta_1, \psi} \qquad \overset{\vdots\, h_{12}}{(\mathsf{K}_a[\mathfrak{u}\cdot q']\psi')_{q' \in F(\mathfrak{u})_a}, \Gamma_1 \Rightarrow \Delta_1, \psi}}{[\mathfrak{u}]\mathsf{K}_a\,\psi', \Gamma_1 \Rightarrow \Delta_1, \psi}\;(\mathsf{L}[]\mathsf{K}) \qquad \overset{\vdots\, h_2}{\psi, \Gamma_2 \Rightarrow \Delta_2}}{[\mathfrak{u}]\mathsf{K}_a\,\psi', \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\;(\mathrm{Cut})$$

is transformed into

$$\dfrac{\dfrac{\overset{\vdots\, h_{11}}{\Gamma_1 \Rightarrow {}^{\cdot}\mathfrak{u}, \Delta_1, \psi} \qquad \overset{\vdots\, h_2}{\psi, \Gamma_2 \Rightarrow \Delta_2}}{\Gamma_1, \Gamma_2 \Rightarrow {}^{\cdot}\mathfrak{u}, \Delta_1, \Delta_2}\;(\mathrm{Cut}) \qquad \dfrac{\overset{\vdots\, h_{12}}{(\mathsf{K}_a[\mathfrak{u}\cdot q']\psi')_{q' \in F(\mathfrak{u})_a}, \Gamma_1 \Rightarrow \Delta_1, \psi} \qquad \overset{\vdots\, h_2}{\psi, \Gamma_2 \Rightarrow \Delta_2}}{(\mathsf{K}_a[\mathfrak{u}\cdot q']\psi')_{q' \in F(\mathfrak{u})_a}, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\;(\mathrm{Cut})}{[\mathfrak{u}]\mathsf{K}_a\,\psi', \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\;(\mathsf{L}[]\mathsf{K})$$

Case 3: In the sequents of both premisses the cut-formula ψ is principal.
Case 3–**S4**: as for $\mathbf{G4}_{P,A}$, see Thm. 1
Case 3–(L[]p)–(R[]p): Then

$$\dfrac{\dfrac{\overset{\vdots\, h_1}{{}^{\cdot}\mathfrak{u}, \Gamma_1 \Rightarrow p, \Delta_1}}{\Gamma_1 \Rightarrow \Delta_1, [\mathfrak{u}]p}\;(\mathrm{R}[]p) \qquad \dfrac{\overset{\vdots\, h_{21}}{\Gamma_2 \Rightarrow {}^{\cdot}\mathfrak{u}, \Delta_2} \qquad \overset{\vdots\, h_{22}}{p, \Gamma_2 \Rightarrow \Delta_2}}{[\mathfrak{u}]p, \Gamma_2 \Rightarrow \Delta_2}\;(\mathrm{L}[]p)}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\;(\mathrm{Cut})$$

is transformed into

$$\dfrac{\dfrac{\dfrac{\overset{\vdots\, h_1}{{}^{\cdot}\mathfrak{u}, \Gamma_1 \Rightarrow p, \Delta_1} \qquad \overset{\vdots\, h_{21}}{\Gamma_2 \Rightarrow {}^{\cdot}\mathfrak{u}, \Delta_2}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, p}\;(\mathrm{Cut}) \qquad \overset{\vdots\, h_{22}}{p, \Gamma_2 \Rightarrow \Delta_2}}{\Gamma_1, \Gamma_2, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \Delta_2}\;(\mathrm{Cut})}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\;(\mathrm{Contr})$$

where $rk({}^{\cdot}\mathfrak{u}) \leq rk(\mathfrak{u}) < rk([\mathfrak{u}]p)$ and $rk(p) = 1 < rk([\mathfrak{u}]p)$.
Case 3–(L[]$\supset$)–(R[]$\supset$): Then

$$\dfrac{\dfrac{\overset{\vdots\, h_1}{[\mathfrak{u}]\psi_1, \Gamma_1 \Rightarrow [\mathfrak{u}]\psi_2, \Delta_1}}{\Gamma_1 \Rightarrow \Delta_1, [\mathfrak{u}](\psi_1 \supset \psi_2)}\;(\mathrm{R}[]\supset) \qquad \dfrac{\overset{\vdots\, h_{21}}{\Gamma_2 \Rightarrow [\mathfrak{u}]\psi_1, \Delta_2} \qquad \overset{\vdots\, h_{22}}{[\mathfrak{u}]\psi_2, \Gamma_2 \Rightarrow \Delta_2}}{[\mathfrak{u}](\psi_1 \supset \psi_2), \Gamma_2 \Rightarrow \Delta_2}\;(\mathrm{L}[]\supset)}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\;(\mathrm{Cut})$$

is transformed into

$$\dfrac{\dfrac{\dfrac{\overset{\vdots\, h_{21}}{\Gamma_2 \Rightarrow [\mathfrak{u}]\psi_1, \Delta_2} \qquad \overset{\vdots\, h_1}{\Gamma_1, [\mathfrak{u}]\psi_1 \Rightarrow \Delta_1, [\mathfrak{u}]\psi_2}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, [\mathfrak{u}]\psi_2}\,(\mathrm{Cut}) \qquad \overset{\vdots\, h_{22}}{[\mathfrak{u}]\psi_2, \Gamma_2 \Rightarrow \Delta_2}}{\Gamma_1, \Gamma_2, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \Delta_2}\,(\mathrm{Cut})}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\,(\mathrm{Contr})$$

where $rk({}^{\cdot}\mathfrak{u}) \leq rk(\mathfrak{u}) < rk([\mathfrak{u}](\psi_1 \supset \psi_2)) = (4 + rk(\mathfrak{u})) \cdot (1 + \max\{rk(\psi_1), rk(\psi_2)\})$ and $rk([\mathfrak{u}]\psi_1), rk([\mathfrak{u}]\psi_2) < rk([\mathfrak{u}](\psi_1 \supset \psi_2))$.
Case 3–(L[]K)–(R[]K): Then

$$\dfrac{\dfrac{\overset{\vdots\, h_{1q'}}{({}^{\cdot}\mathfrak{u}, \Gamma_1 \Rightarrow \mathsf{K}_a[\mathfrak{u} \cdot q']\psi, \Delta_1)_{q' \in F(\mathfrak{u})_a}}}{\Gamma_1 \Rightarrow \Delta_1, [\mathfrak{u}]\mathsf{K}_a\,\psi}\,(\mathsf{R}[]\mathsf{K}) \qquad \dfrac{\overset{\vdots\, h_{21}}{\Gamma_2 \Rightarrow {}^{\cdot}\mathfrak{u}, \Delta_2} \quad \overset{\vdots\, h_{22}}{(\mathsf{K}_a[\mathfrak{u} \cdot q']\psi)_{q' \in F(\mathfrak{u})_a}, \Gamma_2 \Rightarrow \Delta_2}}{[\mathfrak{u}]\mathsf{K}_a\,\psi, \Gamma_2 \Rightarrow \Delta_2}\,(\mathsf{L}[]\mathsf{K})}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\,(\mathrm{Cut})$$

is transformed into

$$\dfrac{\dfrac{\overset{\vdots\, h_{21}}{\Gamma_2 \Rightarrow \Delta_2, {}^{\cdot}\mathfrak{u}} \qquad \dfrac{\dfrac{\overset{\vdots\, h_{1q'_1}}{{}^{\cdot}\mathfrak{u}, \Gamma_1 \Rightarrow \Delta_1, \mathsf{K}_a[\mathfrak{u} \cdot q'_1]\psi} \qquad \dfrac{\dfrac{\overset{\vdots\, h_{1q'_2}}{{}^{\cdot}\mathfrak{u}, \Gamma_1 \Rightarrow \Delta_1, \mathsf{K}_a[\mathfrak{u} \cdot q'_2]\psi} \quad \begin{array}{c} \vdots\, h_{22} \\ (\mathsf{K}_a[\mathfrak{u} \cdot q']\psi)_{q' \in F(\mathfrak{u})_a}, \Gamma_2 \Rightarrow \Delta_2 \\ \vdots\, (\mathrm{Contr}), (\mathrm{Cut}), h_{1q'} \\ (\mathsf{K}_a[\mathfrak{u} \cdot q'_i]\psi)_{i \in \{1,2\}}, {}^{\cdot}\mathfrak{u}, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \end{array}}{\mathsf{K}_a[\mathfrak{u} \cdot q'_1]\psi, {}^{\cdot}\mathfrak{u}, \Gamma_1, {}^{\cdot}\mathfrak{u}, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_1, \Delta_2}\,(\mathrm{Cut})}{\mathsf{K}_a[\mathfrak{u} \cdot q'_1]\psi, {}^{\cdot}\mathfrak{u}, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\,(\mathrm{Contr})}{{}^{\cdot}\mathfrak{u}, \Gamma_1, {}^{\cdot}\mathfrak{u}, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_1, \Delta_2}\,(\mathrm{Cut})}{{}^{\cdot}\mathfrak{u}, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\,(\mathrm{Contr})}{\Gamma_1, \Gamma_2, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \Delta_2}\,(\mathrm{Cut})}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\,(\mathrm{Contr})$$

where the reasoning for $h_{1q'}$ is iterated over all $q' \in F(\mathfrak{u})_a$ as exemplified for $h_{1q'_1}$ and $h_{1q'_2}$. It holds that $rk({}^{\cdot}\mathfrak{u}) \leq rk(\mathfrak{u}) < rk([\mathfrak{u}]\mathsf{K}_a\,\psi) = (4 + rk(\mathfrak{u})) \cdot (1 + rk(\psi))$ and $rk(\mathsf{K}_a[\mathfrak{u} \cdot q']\psi) = 1 + (4 + rk(\mathfrak{u})) \cdot rk(\psi) < rk([\mathfrak{u}]\mathsf{K}_a\,\psi)$.
Case 3–(L[][])–(R[][]): Then

$$\dfrac{\dfrac{\overset{\vdots\, h_1}{\Gamma_1 \Rightarrow \Delta_1, [\mathfrak{u}_1; \mathfrak{u}_2]\psi}}{\Gamma_1 \Rightarrow \Delta_1, [\mathfrak{u}_1][\mathfrak{u}_2]\psi}\,(\mathsf{R}[][]) \qquad \dfrac{\overset{\vdots\, h_2}{[\mathfrak{u}_1; \mathfrak{u}_2]\psi, \Gamma_2 \Rightarrow \Delta_2}}{[\mathfrak{u}_1][\mathfrak{u}_2]\psi, \Gamma_2 \Rightarrow \Delta_2}\,(\mathsf{L}[][])}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\,(\mathrm{Cut})$$

is transformed into

$$\dfrac{\overset{\vdots\, h_1}{\Gamma_1 \Rightarrow \Delta_1, [\mathfrak{u}_1; \mathfrak{u}_2]\psi} \qquad \overset{\vdots\, h_2}{[\mathfrak{u}_1; \mathfrak{u}_2]\psi, \Gamma_2 \Rightarrow \Delta_2}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\,(\mathrm{Cut})$$

where $rk([\mathfrak{u}_1; \mathfrak{u}_2]\psi) < rk([\mathfrak{u}_1][\mathfrak{u}_2]\psi)$. □

By admissibility of (Contr), see Lem. 6, the admissibility of (Cut) is a direct consequence of Lem. 11:

Theorem 3. (Cut) *is admissible for* $\mathbf{G4}_{P,A}[]$.

Proof. We obtain $\vdash_{\mathbf{G4}_{P,A}[]} \Gamma \Rightarrow \Delta$ from $\vdash_{\mathbf{G4}_{P,A}[]} \Gamma \Rightarrow \Delta, \psi$ and $\vdash_{\mathbf{G4}_{P,A}[]} \psi, \Gamma \Rightarrow \Delta$ as follows:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \psi \qquad \psi, \Gamma \Rightarrow \Delta}{\Gamma, \Gamma \Rightarrow \Delta, \Delta}\,\text{Lem. 11}}{\Gamma \Rightarrow \Delta}\,(\mathrm{Contr})$$

□

Theorem 4. $\mathbf{G4}_{P,A}[]$ *is sound and complete for* $\mathbf{DS4}_{P,A}$.

Proof. For soundness, it suffices to check that each rule of $\mathbf{G4}_{P,A}[]$ is valid in $\mathbf{DS4}_{P,A}$; for completeness, that each axiom of the Hilbert-style axiomatisation Tab. 5 is derivable in $\mathbf{G4}_{P,A}[]$ and that each rule is admissible. Modus ponens (MP) follows from Thm. 3. For the axioms we only show the derivations of (red$\supset$) and (red$\mathsf{K}$); all other case are analogous.

Case (red$\supset$): We have

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\vdots\,\text{Lem. 5}}{\psi_1 \Rightarrow \psi_1, \psi_2} \quad \overset{\vdots\,\text{Lem. 5}}{\psi_1, \psi_2 \Rightarrow \psi_2}}{\psi_1 \supset \psi_2, \psi_1 \Rightarrow \psi_2}\,(\mathrm{L}\supset)}{[\mathfrak{u}](\psi_1 \supset \psi_2), [\mathfrak{u}]\psi_1 \Rightarrow [\mathfrak{u}]\psi_2}\,(\mathrm{LR}[]), \text{Lem. 7}}{[\mathfrak{u}](\psi_1 \supset \psi_2) \Rightarrow [\mathfrak{u}]\psi_1 \supset [\mathfrak{u}]\psi_2}\,(\mathrm{R}\supset) \quad \dfrac{\dfrac{\overset{\vdots\,\text{Lem. 5}}{[\mathfrak{u}]\psi_1 \Rightarrow [\mathfrak{u}]\psi_1, [\mathfrak{u}]\psi_2} \quad \overset{\vdots\,\text{Lem. 5}}{[\mathfrak{u}]\psi_1, [\mathfrak{u}]\psi_2 \Rightarrow [\mathfrak{u}]\psi_2}}{[\mathfrak{u}]\psi_1, [\mathfrak{u}]\psi_1 \supset [\mathfrak{u}]\psi_2 \Rightarrow [\mathfrak{u}]\psi_2}\,(\mathrm{L}\supset)}{[\mathfrak{u}]\psi_1 \supset [\mathfrak{u}]\psi_2 \Rightarrow [\mathfrak{u}](\psi_1 \supset \psi_2)}\,(\mathrm{R}[]\supset)}{\Rightarrow [\mathfrak{u}](\psi_1 \supset \psi_2) \leftrightarrow ([\mathfrak{u}]\psi_1 \supset [\mathfrak{u}]\psi_2)}\,(\mathrm{R}\leftrightarrow)$$

Case (red$\mathsf{K}$): We have

$$\dfrac{\overset{\vdots\,*_1}{[\mathfrak{u}]\mathsf{K}_a\,\psi \Rightarrow {}^{\cdot}\mathfrak{u} \supset \bigwedge_{q' \in F(\mathfrak{u})_a} \mathsf{K}_a[\mathfrak{u} \cdot q']\psi} \quad \overset{\vdots\,*_2}{{}^{\cdot}\mathfrak{u} \supset \bigwedge_{q' \in F(\mathfrak{u})_a} \mathsf{K}_a[\mathfrak{u} \cdot q']\psi \Rightarrow [\mathfrak{u}]\mathsf{K}_a\,\psi}}{\Rightarrow [\mathfrak{u}]\mathsf{K}_a\,\psi \leftrightarrow {}^{\cdot}\mathfrak{u} \supset \bigwedge_{q' \in F(\mathfrak{u})_a} \mathsf{K}_a[\mathfrak{u} \cdot q']\psi}\,(\mathrm{R}\leftrightarrow)$$

with $*_1$ given by

$$\dfrac{\dfrac{\overset{\vdots\,\text{Lem. 5}}{{}^{\cdot}\mathfrak{u} \Rightarrow {}^{\cdot}\mathfrak{u}, \bigwedge_{q' \in F(\mathfrak{u})_a} \mathsf{K}_a[\mathfrak{u} \cdot q']\psi} \quad \dfrac{\overset{\vdots\,\text{Lem. 5}}{\big({}^{\cdot}\mathfrak{u}, (\mathsf{K}_a[\mathfrak{u} \cdot q']\psi)_{q' \in F(\mathfrak{u})_a} \Rightarrow \mathsf{K}_a[\mathfrak{u} \cdot q]\psi\big)_{q \in F(\mathfrak{u})_a}}}{{}^{\cdot}\mathfrak{u}, (\mathsf{K}_a[\mathfrak{u} \cdot q']\psi)_{q' \in F(\mathfrak{u})_a} \Rightarrow \bigwedge_{q' \in F(\mathfrak{u})_a} \mathsf{K}_a[\mathfrak{u} \cdot q']\psi}\,(\mathrm{R}\wedge)^+}{{}^{\cdot}\mathfrak{u}, [\mathfrak{u}]\mathsf{K}_a\,\psi \Rightarrow \bigwedge_{q' \in F(\mathfrak{u})_a} \mathsf{K}_a[\mathfrak{u} \cdot q']\psi}\,(\mathrm{L}[]\mathsf{K})}{[\mathfrak{u}]\mathsf{K}_a\,\psi \Rightarrow {}^{\cdot}\mathfrak{u} \supset \bigwedge_{q' \in F(\mathfrak{u})_a} \mathsf{K}_a[\mathfrak{u} \cdot q']\psi}\,(\mathrm{R}\supset)$$

where $(\mathrm{R}\wedge)^+$ denotes iterated application of $(\mathrm{R}\wedge)$, and $*_2$ is given by

$$\dfrac{\dfrac{\overset{\vdots\,\text{Lem. 5}}{\big({}^{\cdot}\mathfrak{u} \Rightarrow {}^{\cdot}\mathfrak{u}, \mathsf{K}_a[\mathfrak{u} \cdot q]\psi\big)_{q \in F(\mathfrak{u})_a}} \quad \dfrac{\overset{\vdots\,\text{Lem. 5}}{\big({}^{\cdot}\mathfrak{u}, (\mathsf{K}_a[\mathfrak{u} \cdot q']\psi)_{q' \in F(\mathfrak{u})_a} \Rightarrow \mathsf{K}_a[\mathfrak{u} \cdot q]\psi\big)_{q \in F(\mathfrak{u})_a}}}{\big({}^{\cdot}\mathfrak{u}, \bigwedge_{q' \in F(\mathfrak{u})_a} \mathsf{K}_a[\mathfrak{u} \cdot q']\psi \Rightarrow \mathsf{K}_a[\mathfrak{u} \cdot q]\psi\big)_{q \in F(\mathfrak{u})_a}}\,(\mathrm{L}\wedge)^+}{\big({}^{\cdot}\mathfrak{u}, {}^{\cdot}\mathfrak{u} \supset \bigwedge_{q' \in F(\mathfrak{u})_a} \mathsf{K}_a[\mathfrak{u} \cdot q']\psi \Rightarrow \mathsf{K}_a[\mathfrak{u} \cdot q]\psi\big)_{q \in F(\mathfrak{u})_a}}\,(\mathrm{L}\supset)}{{}^{\cdot}\mathfrak{u} \supset \bigwedge_{q' \in F(\mathfrak{u})_a} \mathsf{K}_a[\mathfrak{u} \cdot q']\psi \Rightarrow [\mathfrak{u}]\mathsf{K}_a\,\psi}\,(\mathrm{R}[]\mathsf{K})$$

where $(\mathrm{L}\wedge)^+$ denotes iterated application of $(\mathrm{L}\wedge)$. $\square$

4 Conclusions

We presented the novel ordinary Gentzen-type calculus $\mathbf{G4}_{P,A}[]$ for Dynamic Epistemic Logic. The special feature of $\mathbf{G4}_{P,A}[]$ is that — instead of internalising the accessibility relation — the rules for the action modality correspond to the reduction rules in [26, 5]. The main results of this work are the admissibility of the cut rule and the completeness of the calculus.

Currently, $\mathbf{G4}_{P,A}[]$ is based on **S4** modal logic. In the future, we want to extend our calculus to **S5** and to include rules for general knowledge and with further action combinators like selection and iteration. We also want to integrate the calculus into a systematic software development approach for collective adaptive systems [45].

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Lemma 12. *For all* $p \in P$, $\psi, \psi_1, \psi_2 \in \Psi_{P,A}$, $\mathfrak{u}, \mathfrak{u}_1, \mathfrak{u}_2 \in \mathfrak{U}_{P,A}$, *and* $q \in Q(\mathfrak{u})$ *it holds that*

1. $rk([\mathfrak{u}]p) > rk({}^{\cdot}\mathfrak{u} \supset p)$
2. $rk([\mathfrak{u}](\psi_1 \supset \psi_2)) > rk([\mathfrak{u}]\psi_1 \supset [\mathfrak{u}]\psi_2)$
3. $rk([\mathfrak{u}]\mathsf{K}_a\,\psi) > rk({}^{\cdot}\mathfrak{u} \supset \mathsf{K}_a[\mathfrak{u}]\psi)$
4. $rk([\mathfrak{u}]\mathsf{K}_a\,\psi) > \mathsf{K}_a[\mathfrak{u} \cdot q]\psi$
5. $rk([\mathfrak{u}_1][\mathfrak{u}_2]\psi) > rk([\mathfrak{u}_1;\mathfrak{u}_2]\psi)$

Proof. (1) $rk([\mathfrak{u}]p) = (4 + rk(\mathfrak{u})) \cdot rk(p) = 4 + rk(\mathfrak{u}) >$
$1 + rk({}^{\cdot}\mathfrak{u}) = 1 + \max\{rk({}^{\cdot}\mathfrak{u}), rk(p)\} = rk({}^{\cdot}\mathfrak{u} \supset p)$

(2) $rk([\mathfrak{u}](\psi_1 \supset \psi_2)) = (4 + rk(\mathfrak{u})) \cdot rk(\psi_1 \supset \psi_2) = (4 + rk(\mathfrak{u})) \cdot (1 + \max\{rk(\psi_1), rk(\psi_2)\}) >$
$1 + \max\{(4 + rk(\mathfrak{u})) \cdot rk(\psi_1), (4 + rk(\mathfrak{u})) \cdot rk(\psi_2)\} = 1 + \max\{rk([\mathfrak{u}]\psi_1), rk([\mathfrak{u}]\psi_2)\} =$
$rk([\mathfrak{u}]\psi_1 \supset [\mathfrak{u}]\psi_2)$

(3) $rk([\mathfrak{u}]\mathsf{K}_a\,\psi) = (4 + rk(\mathfrak{u})) \cdot rk(\mathsf{K}_a\,\psi) = (4 + rk(\mathfrak{u})) \cdot (1 + rk(\psi)) >$
$1 + \max\{rk({}^{\cdot}\mathfrak{u}), 1 + (4 + rk(\mathfrak{u})) \cdot rk(\psi)\} = 1 + \max\{rk({}^{\cdot}\mathfrak{u}), 1 + rk([\mathfrak{u}]\psi)\} =$
$1 + \max\{rk({}^{\cdot}\mathfrak{u}), rk(\mathsf{K}_a[\mathfrak{u}]\psi)\} = rk({}^{\cdot}\mathfrak{u} \supset \mathsf{K}_a[\mathfrak{u}]\psi)$

(4) $rk([\mathfrak{u}]\mathsf{K}_a\,\psi) = (4 + rk(\mathfrak{u})) \cdot rk(\mathsf{K}_a\,\psi) = (4 + rk(\mathfrak{u})) \cdot (1 + rk(\psi)) >$
$1 + (4 + rk(\mathfrak{u} \cdot q)) \cdot rk(\psi) = 1 + rk([\mathfrak{u} \cdot q]\psi) = rk(\mathsf{K}_a[\mathfrak{u} \cdot q]\psi)$

(5) $rk([\mathfrak{u}_1][\mathfrak{u}_2]\psi) = (4 + rk(\mathfrak{u}_1)) \cdot rk([\mathfrak{u}_2]\psi) = (4 + rk(\mathfrak{u}_1)) \cdot (4 + rk(\mathfrak{u}_2)) \cdot rk(\psi) >$
$(4 + 1 + \max\{rk(\mathfrak{u}_1), (4 + rk(\mathfrak{u}_1)) \cdot rk(\mathfrak{u}_2)\}) \cdot rk(\psi) =$
$(4 + 1 + \max\big(\{rk({}^{\cdot}(\mathfrak{u}_1 \cdot q_1)) \mid q_1 \in Q(\mathfrak{u}_1)\} \cup$
$\{(4 + rk(\mathfrak{u}_1 \cdot q_1)) \cdot rk({}^{\cdot}(\mathfrak{u}_2 \cdot q_2))\} \mid q_1 \in Q(\mathfrak{u}_1), q_2 \in Q(\mathfrak{u}_2)\}\big)) \cdot rk(\psi) =$
$(4 + \max\{1 + \max\{rk({}^{\cdot}(\mathfrak{u}_1 \cdot q_1)),$
$(4 + rk(\mathfrak{u}_1 \cdot q_1)) \cdot rk({}^{\cdot}(\mathfrak{u}_2 \cdot q_2))\} \mid q_1 \in Q(\mathfrak{u}_1), q_2 \in Q(\mathfrak{u}_2)\}) \cdot rk(\psi) =$
$(4 + \max\{1 + \max\{rk({}^{\cdot}(\mathfrak{u}_1 \cdot q_1)), rk([\mathfrak{u}_1 \cdot q_1]{}^{\cdot}(\mathfrak{u}_2 \cdot q_2))\} \mid q_1 \in Q(\mathfrak{u}_1), q_2 \in Q(\mathfrak{u}_2)\}) \cdot rk(\psi) =$
$(4 + \max\{rk({}^{\cdot}(\mathfrak{u}_1 \cdot q_1) \wedge [\mathfrak{u}_1 \cdot q_1]{}^{\cdot}(\mathfrak{u}_2 \cdot q_2)) \mid q_1 \in Q(\mathfrak{u}_1), q_2 \in Q(\mathfrak{u}_2)\}) \cdot rk(\psi) =$
$(4 + \max\{rk({}^{\cdot}\big((\mathfrak{u}_1 \cdot q_1); (\mathfrak{u}_2 \cdot q_2)\big)) \mid q_1 \in Q(\mathfrak{u}_1), q_2 \in Q(\mathfrak{u}_2)\}) \cdot rk(\psi) =$
$(4 + \max\{rk({}^{\cdot}\big((\mathfrak{u}_1; \mathfrak{u}_2) \cdot (q_1, q_2)\big)) \mid (q_1, q_2) \in Q(\mathfrak{u}_1; \mathfrak{u}_2)\}) \cdot rk(\psi) =$
$(4 + rk(\mathfrak{u}_1; \mathfrak{u}_2)) \cdot rk(\psi) = rk([\mathfrak{u}_1; \mathfrak{u}_2]\psi)$ □